

C^* -ALGEBRAS OF MULTIVARIABLE WIENER-HOPF OPERATORS

BY

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ABSTRACT. The C^* -algebra \mathfrak{B} generated by the Wiener-Hopf operators defined over a subsemigroup of a locally compact group is shown to be the image of a groupoid C^* -algebra under a suitable representation. When the subsemigroup is either a polyhedral cone or a homogeneous, self-dual cone in an Euclidean space, this representation may be used to show that \mathfrak{B} is postliminal and to find a composition series with very explicit subquotients. This yields a concrete parameterization of the spectrum of \mathfrak{B} and exhibits the topology on it.

1. Introduction. Over the last twenty years Banach algebra techniques have been used with spectacular success to study Wiener-Hopf equations; i.e., equations of the form

$$[I + W(f)]\xi = \eta$$

where $f \in L^1(\mathbf{R})$, η and the unknown ξ are in $L^2([0, \infty))$, and where $W(f)\xi$ is defined by the formula

$$(1.1) \quad W(f)\xi(s) = \int_0^\infty f(s-t)\xi(t) dt.$$

The reason for much of the success may be traced to the fact that the C^* -algebra \mathfrak{B} generated by all the $W(f)$ as f runs through $L^1(\mathbf{R})$ has a particularly simple form. Indeed, \mathfrak{B} contains the full algebra \mathcal{K} of compact operators on $L^2([0, \infty))$ as an ideal and the quotient \mathfrak{B}/\mathcal{K} is isomorphic to $C_0(\mathbf{R})$.² During the last ten years, or so, there has been an increasing interest in the problem of finding the structure of C^* -algebras generated by multivariable Wiener-Hopf operators. However, this problem is not entirely well posed. The reason is that not only do the C^* -algebras in question depend on the dimension of the underlying Euclidean space, but they depend also on how the interval $[0, \infty)$ is generalized. From an ecumenical perspective one is led quickly to study C^* -algebras generated by operators of the following very general type: Let P be an arbitrary closed cone in \mathbf{R}^n (which is the closure of its interior) and for $f \in L^1(\mathbf{R}^n)$, define $W(f)$ on $L^2(P)$ by the formula

$$W(f)\xi(s) = \int_P f(s-t)\xi(t) dt, \quad \xi \in L^2(P).$$

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²Ordinarily, one includes the identity in \mathfrak{B} . From our perspective it is preferable not to do this.

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We let $\mathfrak{B}(P)$ be the C^* -algebras generated by these $W(f)$ as f runs over $L^1(\mathbf{R}^n)$. Our objective is to describe $\mathfrak{B}(P)$.

As will be made clear later, the structure of $\mathfrak{B}(P)$ is intimately tied to the facial structure of P . Consequently, for the most part we shall restrict our attention to those cones whose facial structure is most tractable. These turn out to be polyhedral cones and homogeneous, self-dual cones. A polyhedral cone is simply the cone generated by a finite number of points and so there are only a finite number of faces. On the other hand, each homogeneous, self-dual cone may be described as the set of squares in a formally real, finite dimensional Jordan algebra. The so-called quadratic representation effects a bijection between the idempotents in the algebra and the faces of the cone. It will develop that for either type of cone P , $\mathfrak{B}(P)$ is solvable in the following sense: A C^* -algebra A is called *solvable* if it has a finite composition series

$$\{0\} = I_{-1} \subseteq I_0 \subseteq I_1 \subseteq \cdots \subseteq I_N = A$$

such that I_k/I_{k-1} , $k = 0, 1, \dots, N$, is isomorphic to a C^* -algebra of the form $C_0(Z_k) \otimes \mathcal{K}$ where Z_k is a locally compact Hausdorff space and where \mathcal{K} is the algebra of compact operators on some Hilbert space. The smallest N for which such a series exists is called the *length* of A . The terminology is due to Dynin [14]. It follows that A is postliminal and that its spectrum is easy to describe. Assume that the length of A is N . Then as a set, the spectrum of A , \hat{A} , is the disjoint union of the Z_k , $\bigcup_{k=0}^N Z_k$, and a basis for the topology on \hat{A} consists of sets of the form $\mathcal{U} \cup Z_{k-1} \cup \cdots \cup Z_0$ where \mathcal{U} is open in Z_k . (The fact that the topology on \hat{A} has such a description follows from the minimality of N .) For the cones P of the type we can handle, we are able to write down explicitly all of the irreducible representations of $\mathfrak{B}(P)$. We will find that I_0 is the algebra of compact operators on $L^2(P)$; I_{N-1} is the commutator ideal of $\mathfrak{B}(P)$; and Z_N is homeomorphic to \mathbf{R}^n , the space containing P . If P is a polyhedral cone in \mathbf{R}^n we will find that the length of $\mathfrak{B}(P)$ is n , and Z_k is homeomorphic to $\mathfrak{F}_k \times \mathbf{R}^k$ where \mathfrak{F}_k is the set of faces in P of dimension k — \mathfrak{F}_k is a finite set and we give it the discrete topology. (We always count $\{0\}$ and P among the faces of P ; also, $\mathbf{R}^0 = \{0\}$ by convention.)

Our analysis of $\mathfrak{B}(P)$ when P is polyhedral thus yields, with minor variations, the results of Douglas and Howe [13] and Dynin [14] concerning $\mathfrak{B}(P)$ when P is the positive orthant in \mathbf{R}^n , i.e., when $P = [0, \infty) \times \cdots \times [0, \infty)$, n -times. Suppose next that P is a homogeneous, self-dual cone and suppose in addition that P is *irreducible*. This means that P can't be written as a product of lower dimensional cones; this means equivalently that the associated Jordan algebra \mathfrak{A} is simple. In such a case we show that the length of $\mathfrak{B}(P)$ is the degree or rank of \mathfrak{A} ; i.e., the number of linearly independent, minimal idempotents needed to sum to the identity of \mathfrak{A} . The space Z_k , then, turns out to be a certain vector bundle over a Grassmannian built from the idempotents in \mathfrak{A} of degree k . If, in particular, P is the forward light cone in \mathbf{R}^n , $n \geq 3$, i.e., if $P = \{(x_1, \dots, x_n) \mid x_n \geq 0, x_n^2 \geq \sum_{k=1}^{n-1} x_k^2\}$, then $\mathfrak{B}(P)$ is solvable of length 2 and we have: $I_0 = \mathcal{K}$, $I_1/I_0 \cong C_0(S^{n-2} \times \mathbf{R}^{n-1}) \otimes \mathcal{K}$, and $I_2/I_1 \cong C_0(\mathbf{R}^n)$. With the aid of the Cayley transform and the Fourier transform, this result may be seen to complement the work of Berger and Coburn [3] concerning Toeplitz

operators on the group of two-by-two unitary matrices and their more recent, joint work with Koranyi [4] on Wiener-Hopf operators defined on Lie spheres.

We have thus identified a composition series for $\mathfrak{B}(P)$ and we have identified the corresponding subquotients when the cone P belongs to certain special, but important, classes. The next step is to determine the K -theory of $\mathfrak{B}(P)$ and to exploit it for the purpose of deciding invertibility questions for the Wiener-Hopf operators. It is well known that the problem of identifying the K -theory of a general postliminal C^* -algebra is quite difficult. However, due to the explicit and concrete fashion in which the pertinent parameters appear, we are optimistic that for $\mathfrak{B}(P)$ the problem is tractable. We intend to pursue it in future publications.

We arrive at our results by showing how to realize $\mathfrak{B}(P)$ as a faithful representation of the C^* -algebra $C^*(\mathfrak{G})$ of an explicitly constructed groupoid \mathfrak{G} . (A concise accounting of the theory of groupoids and groupoid C^* -algebras will be given in §2.) We then use the theory of such C^* -algebras developed by the second author in [25] to read off the structure of $\mathfrak{B}(P)$. The groupoid \mathfrak{G} is constructed from P as follows. Let Y be the maximal ideal space of the C^* -algebra generated by $1_P * f$ as f runs through $L^1(\mathbf{R}^n)$. Then assuming $P \cap -P = \{0\}$, \mathbf{R}^n is “contained” in Y and \mathbf{R}^n acts on Y in a natural way determining a locally compact transformation group. When $n = 1$, and $P = [0, \infty)$, it is easy to see that Y is homeomorphic to $(-\infty, \infty]$ and that \mathbf{R} acts by translation on $(-\infty, \infty)$ while leaving ∞ fixed. This transformation group and related transformation groups have appeared in the analysis of Wiener-Hopf operators before (cf. [8, 9, 12]), but the deeper parts of the theory of transformation group of C^* -algebras have not been exploited. The reason, no doubt, is due in part to the fact that the transformation group (Y, \mathbf{R}^n) is not precisely what one wants. If one views (Y, \mathbf{R}^n) as a groupoid and if one lets X be the closure of P in Y , then what one wants is the groupoid \mathfrak{G} obtained by reducing $Y \times \mathbf{R}^n$ to X . As a space, $\mathfrak{G} = \{(x, t) \mid x \in X, t \in \mathbf{R}^n \text{ and } x + t \in X\}$ with the obvious topology. The theory of groupoid C^* -algebras generalized the theory of transformation group C^* -algebras in many ways and in particular, as with transformation groups, the ideals in a groupoid C^* -algebra are parameterized with the aid of the orbit structure of the groupoid. The orbits for our \mathfrak{G} turn out to be the intersections of X with the orbits of Y . In the one variable setting it turns out that the orbits are the sets $[0, \infty)$ and $\{\infty\}$. The orbit $[0, \infty)$ has no isotropy and so the ideal it determines is, as we shall see, \mathcal{K} , while the isotropy group of ∞ is \mathbf{R} . Since $C^*(\mathbf{R}) \cong C_0(\mathbf{R})$ we rearrive at the classical fact concerning the structure of the C^* -algebra of Wiener-Hopf operators on the half-line. (All this will be clarified in §§2 and 3.) It thus results that the main task of our analysis is to identify the orbits of Y .

In a sense we have left begging an obvious and important question: How does the ingenuous investigator, one who is uninitiated to the theory of groupoid C^* -algebras, automatically find groupoids thrust upon him? The answer is actually very easy. As soon as it is recognized that generally $W(f_1 * f_2) \neq W(f_1)W(f_2)$, it is natural to seek a larger algebra of functions, containing L^1 as a linear subspace, to which W extends as a representation. Of course such a ploy is not entirely new. To a certain extent it has already appeared in earlier studies of Wiener-Hopf operators (cf. [20]) and it has

been used with considerable success in the theory of pseudo-differential operators (cf. [16]). Our contribution is to make explicit the role of groupoids in the analysis of these extension algebras. Let's see how things work out in the one-dimensional setting.

A change of variable converts equation (1.1) to the expression $W(f)\xi(t) = \int_{-\infty}^{\infty} f(-s)1_{[0,\infty)}(t+s)\xi(t+s)ds$. It is convenient to suppress the minus sign and simply to redefine $W(f)$ by the formula

$$(1.2) \quad W(f)\xi(t) = \int_{-\infty}^{\infty} f(s)1_{[0,\infty)}(t+s)\xi(t+s)ds, \quad \xi \in L^2[0, \infty).$$

This has no effect on \mathfrak{B} . Now calculate the product $W(f_1)W(f_2)$, $f_1, f_2 \in L^1(\mathbf{R})$, to find that

$$W(f_1)W(f_2)\xi(t) = \int_{-\infty}^{\infty} h(t, s)1_{[0,\infty)}(t+s)\xi(t+s)ds$$

where $h(t, s) = \int_{-\infty}^{\infty} f_1(r)1_{[0,\infty)}(t+r)f_2(s-r)dr$. Next take four functions $f_i \in L^1(\mathbf{R})$, $i = 1, 2, 3, 4$, let h_1 correspond to $W(f_1)W(f_2)$ and let h_2 correspond to $W(f_3)W(f_4)$. Then we have

$$(1.3) \quad (W(f_1)W(f_2)W(f_3)W(f_4))\xi(t) = \int_{-\infty}^{\infty} h(t, s)1_{[0,\infty)}(t+s)\xi(t+s)ds$$

where

$$(1.4) \quad h(t, s) = \int_{-\infty}^{\infty} h_1(t, r)1_{[0,\infty)}(t+r)h_2(t+r, s-r)dr.$$

One is thus led to consider the collection \mathfrak{F}_0 of functions h on $[0, \infty) \times \mathbf{R}$ with the property that

$$\text{ess sup}_t \int_{-\infty}^{\infty} |h(t, s)|ds$$

is finite. With respect to the product defined by (1.4) and this expression as norm, \mathfrak{F}_0 becomes a Banach *-algebra where we set $h^*(t, s) = 1_{[0,\infty)}(t+s)\overline{h(t+s, -s)}$ by definition. An f in $L^1(\mathbf{R})$ is identified with an element in \mathfrak{F}_0 in the obvious way and formula (1.3) then gives the extension of W to all of \mathfrak{F}_0 . The algebra \mathfrak{F}_0 is much too big in two respects. First of all the closure of $W(\mathfrak{F}_0)$ is larger than \mathfrak{B} . This can be remedied by cutting \mathfrak{F}_0 down to the smallest subalgebra generated by $L^1(\mathbf{R})$. More importantly, W is not faithful on \mathfrak{F}_0 . A little experimentation reveals that the collection \mathfrak{F} consisting of those functions on $[0, \infty) \times \mathbf{R}$ which are supported on $\mathfrak{G} = \{(t, s) | t+s \geq 0\}$ is a subalgebra and that the kernel of W consists of those functions which are supported on the complement of this set. But from what we've seen before, \mathfrak{G} is just the reduction to $[0, \infty)$ of the transformation group determined by \mathbf{R} acting on \mathbf{R} . One could push our reasoning further to deduce the multiplication on \mathfrak{G} , to deduce that it's necessary to append the point at infinity to $[0, \infty)$, and to discover other facts pertinent to the theory. However, we shall stop here, having put \mathfrak{G} into evidence on the basis of naive reasoning, and begin de novo with our analysis.

In the next section we present those aspects of the theory of groupoid C^* -algebras which we need. Our presentation will be self-contained for the benefit of the reader,

but it will be brief since it is condensed almost entirely from [25], except for some important new technical results. While the reader may well find that the groupoids encountered in this paper are either familiar or so related to familiar groupoids that a development of the theory of arbitrary groupoid C^* -algebras may not seem warranted, we hope that a careful reading will reveal that an effort to avoid totally the theory of groupoids would be wasteful. It would require just as much space as that used in the present presentation to set up and analyze the structures we need to manipulate throughout the entire paper. Moreover, without the use of groupoids, a large portion of the analysis would seem ad hoc and unmotivated. It seems to us that much of the power in the groupoid approach to C^* -algebras lies in the unifying perspective it provides for studying certain types of operations on C^* -algebras—operations which may be familiar in special cases, but which, in fact, can be applied quite broadly. In §3, we discuss the construction of the groupoid and groupoid C^* -algebra for the C^* -algebra of Wiener-Hopf operators defined over a subsemi-group of an arbitrary locally compact group. We conduct our analysis with such great generality because we have an eye toward future applications and because it clarifies the role of each hypothesis as it is made. The analysis of $\mathfrak{B}(P)$, when P is a polyhedral cone, is presented in §4. §5 is a respite in which we gather material from the theory of Jordan algebras necessary to our discussion. Again, it will be self-contained for the benefit of the reader, but brief. Finally, our analysis of $\mathfrak{B}(P)$, when P is a self-dual, homogeneous cone, is presented in §6.

We adopt standard notation and conventions. All Hilbert spaces will be complex and separable and all operators will be bounded and linear. The algebra of all operators on a Hilbert space \mathcal{H} will be denoted either by \mathfrak{L} or by $\mathfrak{L}(\mathcal{H})$, while the ideal of compact operators will be denoted by \mathfrak{K} or $\mathfrak{K}(\mathcal{H})$. All locally compact spaces considered will be second countable and all measures on such spaces are assumed to be Radon measures and nonnegative.

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2. Groupoids and groupoid C^* -algebras.

2.1 By definition a *groupoid* is a set \mathfrak{G} together with a pair of mappings satisfying the following axioms. The domain of the first mapping is a subset $\mathfrak{G}^2 \subseteq \mathfrak{G} \times \mathfrak{G}$ called the set of *composable pairs* and the image of (x, y) in \mathfrak{G}^2 is denoted xy . The second mapping is an involution of \mathfrak{G} and the image of x in \mathfrak{G} under this map is denoted by x^{-1} . The axioms are

- (i) $(x, y), (y, z) \in \mathfrak{G}^2 \Rightarrow (xy, z), (x, yz) \in \mathfrak{G}^2$ and $(xy)z = x(yz)$,
- (ii) $(x^{-1}, x), (x, x^{-1}) \in \mathfrak{G}^2$ for all x in \mathfrak{G}^2 ,
- (iii) $(x, y), (z, x) \in \mathfrak{G} \Rightarrow x^{-1}(xy) = y$ and $(zx)x^{-1} = z$.

If, in addition, \mathfrak{G} is a locally compact Hausdorff space and if the maps are continuous (we give \mathfrak{G}^2 the relative topology) then \mathfrak{G} is called a *locally compact groupoid*. The maps d and r from \mathfrak{G} to \mathfrak{G} defined by the formulas $d(x) = x^{-1}x$ and $r(x) = xx^{-1}$ are called the *domain* and *range* maps respectively. They have a common image \mathfrak{G}^0 called the *unit space* of \mathfrak{G} . Evidently, a pair (x, y) belongs to \mathfrak{G}^2 if and only if $d(x) = r(y)$. Since r and d are continuous when \mathfrak{G} is a locally compact

groupoid, it follows that \mathcal{G}^2 is closed in \mathcal{G} . Also, since an element u is a unit, i.e., $u \in \mathcal{G}^0$, if and only if $(u, u) \in \mathcal{G}^2$ and $u^2 = u$, it follows that \mathcal{G}^0 is closed in \mathcal{G} and that the relative topology in \mathcal{G}^0 coincides with the quotient topology.

Since the only groupoids we discuss are locally compact, we drop the adjective “locally compact” from now on.

If \mathcal{G} is a groupoid and $u \in \mathcal{G}^0$, then $d^{-1}(u) \cap r^{-1}(u)$ is a closed subset of \mathcal{G} which has the structure of a locally compact group with identity u . This group is called the *isotropy group* at u or of u . We say that two points u and v in \mathcal{G}^0 lie in the same *orbit* if and only if $d^{-1}(u) \cap r^{-1}(v) \neq \emptyset$. For a fixed u , the orbit through u is $\{v \in \mathcal{G}^0 \mid r^{-1}(v) \cap d^{-1}(u) \neq \emptyset\}$. Evidently the orbits partition \mathcal{G}^0 . A set which is the union of orbits is called *invariant*. The groupoid \mathcal{G} is called *transitive* if and only if there is only one orbit; \mathcal{G} is called *principal* if and only if the isotropy group at each u in \mathcal{G}^0 is trivial. Thus \mathcal{G} is principal if and only if the map $x \rightarrow (r(x), d(x))$ is one-to-one.

2.2 Apart from groups which obviously are groupoids, the two simplest examples of groupoids are constructed from a locally compact space X as follows.

2.2.1 The first, which we call the *trivial groupoid* determined by X , is $\mathcal{G} = X \times X$. The set of composable pairs is $\mathcal{G}^2 = \{(x, y), (x_1, y_1) \mid x_1 = y\}$ and the product of two such pairs is (x, y_1) ; the inverse of (x, y) is (y, x) . It is clear that $r(x, y) = (x, x)$ while $d(x, y) = (y, y)$. Thus the unit space of \mathcal{G} is the diagonal of $X \times X$ which, in turn, may be identified with X .

2.2.2 The other groupoid built from X is called the *cotrivial groupoid* (determined by X). In this case $\mathcal{G} = X$ while $\mathcal{G}^2 = \{(x, y) \mid x = y\}$; i.e., \mathcal{G}^2 is the diagonal of $X \times X$. Two pairs are composable if and only if they are equal and each pair is its own inverse. Both the trivial and the cotrivial groupoids on X are principal, while the trivial one is transitive and the orbits of the cotrivial groupoid are just the points in X .

2.2.3 Actually, these two examples are simply extreme cases of the following construction. Let X be a locally compact space and let R be the graph of an equivalence relation on X . If we assume that R is a locally closed subset of $X \times X$, then R becomes a groupoid \mathcal{G} where $\mathcal{G}^2 = \{(x, y), (x_1, y_1) \mid (x, y) \in R \times R, y = x_1\}$. The product is as before. On the other hand, if \mathcal{G} is a principal groupoid, then the map $x \rightarrow (d(x), r(x))$ sets up an isomorphism between \mathcal{G} and an equivalence relation of $\mathcal{G}^0 \times \mathcal{G}^0$ viewed as a groupoid.

2.2.4 Much of the terminology derives from the theory of transformation groups. Let Y be a locally compact Hausdorff space on which a locally compact group G acts continuously. We write the action of G on Y on the right; i.e., if $y \in Y$ and $t \in G$, then yt denotes the translate of y by t . If G is commutative, we write $y + t$ instead of yt . The set $\mathcal{G} = Y \times G$ becomes a groupoid when we set $\mathcal{G}^2 = \{(y_1, t_1), (y_2, t_2) \mid y_2 = y_1 t_1\}$, defining the product of such a pair to be $(y_1, t_1 t_2)$, and when we define $(y, t)^{-1}$ to be (yt, t^{-1}) . The maps d and r satisfy the equations $d(y, t) = (yt, e)$ and $r(y, t) = (y, e)$ where e is the identity of G . Thus the unit space of \mathcal{G} may be identified with Y . The notions of orbit, isotropy, invariant, etc., in the groupoid setting reduce to the familiar concepts from group actions. We note, in particular, that $\mathcal{G} = Y \times G$ is principal if and only if the action of G on Y is free.

2.2.5 One particular construction which is very important for our purposes is that of reducing or contracting a groupoid to a subset of its unit space. Let \mathcal{G} be a groupoid and let E be a locally closed subset of \mathcal{G}^0 . Then the *reduction* (or *contraction*) of \mathcal{G} to E , denoted $\mathcal{G}|E$, is defined to be $\{x \in \mathcal{G} \mid r(x) \in E, d(x) \in E\}$. It is important to keep in mind that E need not be invariant. It is clear that $\mathcal{G}|E$ is a groupoid with unit space E . If \mathcal{G} is the groupoid in 2.2.4, then $\mathcal{G}|E = \{(x, t) \mid x \in E, xt \in E\}$.

2.3 EXAMPLE. Let $Y = (-\infty, \infty]$, $G = \mathbf{R}$, let \mathbf{R} act by translation on $(-\infty, \infty)$ and let ∞ remain fixed. The transformation group (Y, \mathbf{R}) determines a groupoid $\mathcal{G} = Y \times \mathbf{R}$ which we just described. If $X = [0, \infty]$, then $\mathcal{G}|X = \{(y, s) \in \mathcal{G} \mid y \geq 0, y + s \geq 0\}$, the groupoid we associated with \mathfrak{B} in the introduction. The orbits in X are $[0, \infty)$ and $\{\infty\}$. The isotropy group of ∞ is \mathbf{R} while $\mathcal{G}|[0, \infty) = (\mathcal{G}|X)|[0, \infty)$ is principal and transitive. It is isomorphic to the trivial groupoid on $[0, \infty)$ under the map $(y, t) \rightarrow (r(y, t), d(y, t)) = (y, y + t)$.

2.4 Let \mathcal{G} be a groupoid and let $C_c(\mathcal{G})$ be the space of compactly supported, continuous, complex-valued functions defined on \mathcal{G} . As is customary, $C_c(\mathcal{G})$ is given the inductive limit topology. A family of measures on \mathcal{G} indexed by the unit space of \mathcal{G} , $\{\lambda^u \mid u \in \mathcal{G}^0\}$, is called a (left) *Haar system* on \mathcal{G} in case (i) each λ^u is supported by $r^{-1}(u)$, i.e., $\text{supp}(\lambda^u) = r^{-1}(u)$, (ii) for each $f \in C_c(\mathcal{G})$, the function $u \rightarrow \int f d\lambda^u$ is continuous on \mathcal{G}^0 , and (iii) for each $x \in \mathcal{G}$ and $f \in C_c(\mathcal{G})$, $\int f(xy) d\lambda^{d(x)}(y) = \int f(y) d\lambda^{r(x)}(y)$. In contrast to the theory of locally compact groups, it is not known if a general groupoid carries a left Haar system. Moreover, in most situations, when a Haar system exists, it is highly nonunique in the sense that the two Haar systems need not be proportional. For many purposes, however, the nonuniqueness of Haar systems causes no problems. This is due in part to the fact that two Haar systems give rise to strongly Morita equivalent C*-algebras, at least if a technical requirement is satisfied [25, Chapter II, Corollary 2.11]. It is not known, however, if the C*-algebras are always isomorphic. Since this will be important in our considerations, we will continually have to specify which Haar system is being considered. Usually, however, there will be only one natural one in sight.

2.5 Here are some examples. If \mathcal{G} is the cotrivial groupoid determined by a locally compact space X , then $\mathcal{G} = \mathcal{G}^0 = X$ and the equation $\lambda^x = \delta_x$ defines a left Haar system on \mathcal{G} . If \mathcal{G} is the trivial groupoid determined by X , then identifying \mathcal{G}^0 with X and taking a fixed measure α on X with $\text{supp}(\alpha) = X$, we get a left Haar system $\{\lambda^x \mid x \in X\}$ on \mathcal{G} by setting $\lambda^x = \delta_x \times \alpha$. It is clear from this that Haar systems aren't unique in general. If (Y, G) is a transformation group and $\mathcal{G} = Y \times G$, we may take $\lambda^u = \delta_u \times \lambda$, $u \in Y = \mathcal{G}^0$, where λ is a fixed left Haar measure for G , and obtain a left Haar system $\{\lambda^u \mid u \in Y\}$ on \mathcal{G} . Finally note that if $\{\lambda^u \mid u \in \mathcal{G}^0\}$ is a left Haar system on \mathcal{G} and if $E \subseteq \mathcal{G}^0$, then $\{1_{\mathcal{G}|E} \lambda^u \mid u \in E\}$ is a left Haar system on $\mathcal{G}|E$ provided $1_{\mathcal{G}|E} \lambda^u$, $u \in E$, has support equal to $r^{-1}(u) \cap \mathcal{G}|E$. (We note that such a proviso is necessary because, for example, if \mathcal{G} is the transformation group $\mathbf{R} \times \mathbf{R}$ with Haar system $\lambda^u = \delta_u \times \lambda$ and if $E = \mathbf{Z}$, then $\mathcal{G}|E$ is $\mathbf{Z} \times \mathbf{Z}$ which has measure zero with respect to each λ^u . On the other hand, if E is open or, more generally, if E is the closure of its interior, then the proviso is satisfied.)

2.6 With each Haar system $\{\lambda^u \mid u \in \mathbb{G}^0\}$ on a groupoid \mathbb{G} , one can define a multiplication and involution with respect to which $C_c(\mathbb{G})$ becomes a topological *-algebra [25, Chapter II, Proposition 1.1]. The product is given by the formula

$$f * g(x) = \int f(xy)g(y^{-1}) d\lambda^{d(x)}(y),$$

$f, g \in C_c(\mathbb{G})$, and the involution is given by the formula $f^*(x) = \overline{f(x^{-1})}$.

2.6.1 When \mathbb{G} is the cotrivial groupoid based on a locally compact space X and $\{\lambda^x \mid x \in \mathbb{G}^0\}$ is the associated Haar system (2.5), $C_c(\mathbb{G})$ is nothing but the space of compactly supported continuous functions on \mathbb{G} with the usual involution and pointwise operations.

2.6.2 When \mathbb{G} is the trivial groupoid based on X and $\{\lambda^x \mid x \in \mathbb{G}^0\}$ is the Haar system $\lambda^x = \delta_x \times \alpha$ (2.5), we find that since $d(x, y) = y$ while $r(x, y) = x$ (2.2), the multiplication on $C_c(\mathbb{G})$ is given by the formula

$$f * g(x, y) = \int_X f(x, z)g(z, y) d\alpha(z)$$

and the involution is given by the formula $f^*(x, y) = \overline{f(y, x)}$. Thus $C_c(\mathbb{G})$ consists of “continuous matrices” or integral operators (although we have not yet represented $C_c(\mathbb{G})$ anywhere).

2.6.3 If a locally compact group G , with left Haar measure λ , acts on a locally compact Hausdorff space Y , and if we endow the groupoid $\mathbb{G} = Y \times G$ with the left Haar system $\{\lambda^y \mid y \in Y = \mathbb{G}^0\}$ where $\lambda^y = \delta_y \times \lambda$, then the product of two functions f and $g \in C_c(\mathbb{G})$ is given by the formula

$$\begin{aligned} f * g(x, t) &= \int f((x, t)(y, s))g((y, x)^{-1}) d\lambda^{d(x, t)}(y, s) \\ &= \int f(x, ts)g(xts, s^{-1}) d\lambda(s) \\ &= \int f(x, s)g(xs, s^{-1}t) d\lambda(s), \end{aligned}$$

while $f^*(x, s) = \overline{f(xs, s^{-1})}$. Thus except for the absence of the modular function in the definition of the involution, we recapture the algebraic structure first placed on $C_c(Y \times G)$ by Glimm [18] (cf. [10] and [15] too).

2.6.4 If $\mathbb{G} = Y \times G$ with the Haar system $\lambda^x = \delta_x \times \lambda$ and if E is a locally closed subset of Y such that $1_{\mathbb{G}|E}\lambda^x, x \in E$, has support equal to $r^{-1}(x) \cap \mathbb{G} \mid E$, then direct calculation shows that the product on $C_c(\mathbb{G} \mid E)$ takes the form

$$f * g(x, t) = \int f(x, s)g(xs, s^{-1}t)1_E(xs) d\lambda(s), \quad f, g \in C_c(\mathbb{G} \mid E).$$

2.7 There is another algebra which plays a role in the theory. Let \mathbb{G} be a groupoid with left Haar system $\{\lambda^u \mid u \in \mathbb{G}^0\}$ and on $C_c(\mathbb{G})$, define the norm

$$\|f\|_I = \max \left\{ \sup_{u \in \mathbb{G}^0} \int |f| d\lambda^u, \sup_{u \in \mathbb{G}^0} \int |f^*| d\lambda^u \right\}.$$

It is not difficult to see that with this norm and the operations of 2.6, $C_c(\mathfrak{G})$ is a normed *-algebra whose completion is denoted by $L^1(\mathfrak{G})$ or $L^1(\mathfrak{G}, \lambda)$ if it is necessary to put $\{\lambda^u \mid u \in \mathfrak{G}^0\}$ into evidence (cf. [25, Chapter II, Proposition 1.4]). The enveloping C*-algebra of $L^1(\mathfrak{G})$ is denoted by $C^*(\mathfrak{G})$ or $C^*(\mathfrak{G}, \lambda)$ and is called the groupoid C*-algebra of \mathfrak{G} (and λ).

We note that when $\mathfrak{G} = Y \times G$ as in 2.6.3, then $L^1(\mathfrak{G})$ is closely related to, but different from, an L^1 -algebra which intervenes in the theory of transformation group C*-algebras [15] and, more generally, in the theory of C*-crossed products [28]. Also we mention here, although it takes a little proof, that $C^*(\mathfrak{G})$ coincides with the usual transformation group C*-algebra bases on (Y, G) .

2.7.1 One can see without difficulty that if \mathfrak{G} is the cotrivial groupoid built from a locally compact space X , then $C^*(\mathfrak{G})$ is simply $C_0(X)$. On the other hand, $C^*(\mathfrak{G})$ is an elementary C*-algebra when \mathfrak{G} is the trivial groupoid built from X . If, in particular, X is infinite, $C^*(\mathfrak{G})$ is isomorphic to the C*-algebra of compact operators on an infinite dimensional Hilbert space (cf. [25, Chapter II, Example 2.12]). We will see this again in 2.17.1 below.

2.8 EXAMPLE 2.3 CONTINUED. A Haar system for $\mathfrak{G} \mid X$ is $\{1_{\mathfrak{G} \mid X} \lambda^u \mid u \in X\}$ where $\lambda^u = \delta_u \times \lambda$, $u \in X$, and λ is Lebesgue measure on \mathbf{R} . If we apply 2.6.4 we find that

$$f * g(x, t) = \begin{cases} \int_{-\infty}^{\infty} f(x, s)g(x + s, t - s)1_{[0, \infty)}(x + s) ds, & 0 \leq x < \infty, \\ \int_{-\infty}^{\infty} f(\infty, s)g(\infty, t - s) ds, & x = \infty, \end{cases}$$

for all $f, g \in C_c(\mathfrak{G} \mid X)$. Thus, for all intents and purposes, we have the multiplication on \mathfrak{G} . In fact it is not hard to see, with the aid of Proposition 3.5, that $L^1(\mathfrak{G} \mid X)$ is isometrically *-isomorphic to the subalgebra of \mathfrak{G} generated by the functions \tilde{f} , $f \in L^1(\mathbf{R})$, where $\tilde{f}(x, t) = f(t)$, $(x, t) \in \mathfrak{G} \mid [0, \infty)$. Moreover, as we shall see, equation (1.3) determines a faithful representation of $C^*(\mathfrak{G} \mid X)$ whose image is the C*-algebra generated by the Wiener-Hopf operators.

2.9 For $i = 1, 2$, let $(\mathfrak{G}_i, \lambda_i)$ be a groupoid with left Haar system $\{\lambda_i^u \mid u \in \mathfrak{G}_i^0\}$. A *topological isomorphism* from $(\mathfrak{G}_1, \lambda_1)$ onto $(\mathfrak{G}_2, \lambda_2)$ is an algebraic isomorphism Φ from \mathfrak{G}_1 onto \mathfrak{G}_2 which is also a homeomorphism carrying λ_1^u to $\lambda_2^{\Phi(u)}$. We regard this terminology as somewhat provisional because at this time it is not entirely clear what the “correct” notion of isomorphism is. For example there is a materially weaker notion that has proved useful; one requires only that Φ be a Borel measurable, algebraic isomorphism from \mathfrak{G}_1 onto \mathfrak{G}_2 which carries λ_1^u onto $\lambda_2^{\Phi(u)}$ so that the restriction of Φ to \mathfrak{G}_1^0 is a homeomorphism onto \mathfrak{G}_2^0 . Such an isomorphism might well be called a Borel isomorphism. In any event, the only isomorphisms to appear in this paper are of the topological variety. Note, incidentally, that the isomorphism in Example 2.3 from $\mathfrak{G} \mid [0, \infty)$ to the trivial groupoid on $[0, \infty)$ is a topological isomorphism when each groupoid is given the obvious Haar system.

2.10 PROPOSITION. For $i = 1, 2$, let $(\mathfrak{G}_i, \lambda_i)$ be a groupoid with Haar system and let Φ be a topological isomorphism from \mathfrak{G}_1 onto \mathfrak{G}_2 . Then the map

$$f \rightarrow f \circ \Phi^{-1}$$

from $C_c(\mathbb{G}_1)$ to $C_c(\mathbb{G}_2)$ extends to a C^* -isomorphism from $C^*(\mathbb{G}_1, \lambda_1)$ onto $C^*(\mathbb{G}_2, \lambda_2)$.

PROOF. It suffices to note that this map effects an isometric $*$ -isomorphism between $L^1(\mathbb{G}_1, \lambda_1)$ and $L^1(\mathbb{G}_2, \lambda_2)$.

2.11 Let \mathbb{G} be a groupoid with Haar system $\{\lambda^u \mid u \in \mathbb{G}^0\}$ and let μ be a positive Radon measure on \mathbb{G}^0 . Then μ induces two measures ν and ν^{-1} on \mathbb{G} according to these formulas: $\nu = \int_{\mathbb{G}^0} \lambda^u d\mu(u)$; i.e., $\int f d\nu = \int_{\mathbb{G}^0} \int_{\mathbb{G}^u} f(x) d\lambda^u(x) d\mu(u)$, for all $f \in C_c(\mathbb{G})$; and ν^{-1} is the image of ν under the map $x \rightarrow x^{-1}$; i.e., $\int f(x) d\nu^{-1} = \int f(x^{-1}) d\nu, f \in C_c(\mathbb{G})$.

2.12 The Hilbert space $L^2(\nu^{-1})$ carries a representation of $C^*(\mathbb{G})$ which is called the representation *induced off the unit space by μ* and is denoted by $\text{Ind } \mu$. It is defined by the formula

$$\text{Ind } \mu(f)\xi(x) = \int f(xy)\xi(y^{-1}) d\lambda^{d(x)}(y),$$

$f \in C_c(\mathbb{G})$, $\xi \in L^2(\nu^{-1})$. One checks easily that $\|\text{Ind } \mu(f)\| \leq \|f\|_{L^1}$ so that $\text{Ind } \mu$ extends to all of $C^*(\mathbb{G})$.

2.12.1 Suppose that $\mathbb{G} = Y \times G$ with left Haar system as in 2.6.3. If μ is a measure on $Y = \mathbb{G}^0$, then $\nu = \mu \times \lambda$. If $f \in C_c(\mathbb{G})$ and $\xi \in L^2(\nu^{-1})$, then the calculations in 2.6.3 show that

$$\text{Ind } \mu(f)\xi(x, t) = \int f(x, s)\xi(xs, s^{-1}t) d\lambda(s).$$

On the other hand, the map J defined by the formula $J\xi(x, t) = \xi(xt, t^{-1})$ determines a Hilbert space isomorphism from $L^2(\nu)$ onto $L^2(\nu^{-1})$ with inverse defined by the same formula. Consequently, we discover that for $\xi \in L^2(\nu)$ and $f \in C_c(\mathbb{G})$, $(J^{-1}\text{Ind } \mu(f)J)\xi(x, t) = \int f(xt, s)\xi(x, ts) d\lambda(s)$. Thus, except for the missing modular function, we see that J implements a unitary equivalence between $\text{Ind } \mu$ and the usual representation of $C^*(\mathbb{G})$ which is induced by the representation of $C_0(Y)$ as multiplication operators on $L^2(\mu)$ (cf. [15, 26]). This explains the terminology and notation. Actually, the entire theory of induced representations can be formulated in the context of groupoid C^* -algebras as is shown in §2 in Chapter II of [25]. However, we need only a very small portion of it and so proceed with the pedestrian approach we've been taking.

2.12.2 Suppose that $\mathbb{G} = Y \times G$, that E is a locally closed subset of Y , and that $\mathbb{G}|E$ is given the usual left Haar system. We assume that the proviso in 2.5 is satisfied. If μ is a measure on E , then by 2.6.4, we find that

$$\text{Ind } \mu(f)\xi(x, t) = \int f(x, s)\xi(xs, s^{-1}t)1_E(xs) d\lambda(s),$$

$f \in C_c(\mathbb{G}|E)$, $\xi \in L^2(\nu^{-1})$. The map J defined in 2.12.1 maps $L^2(\nu)$ isometrically onto $L^2(\nu^{-1})$ and is formally self-inverse. Consequently we find that

$$(J^{-1}\text{Ind } \mu(f)J)\xi(x, t) = \int f(xt, s)\xi(x, ts)1_E(xts) d\lambda(s),$$

for all $f \in C_c(\mathbb{G}|E)$ and $\xi \in L^2(\nu)$.

2.13 Suppose that \mathcal{G} is a groupoid with Haar system $\{\lambda^u \mid u \in \mathcal{G}^0\}$ and let μ be a measure on \mathcal{G}^0 . Then if ν is the induced measure on \mathcal{G} , $\nu = \int_{\mathcal{G}^0} \lambda^u d\mu(u)$, we have an obvious and canonical identification of $L^2(\nu^{-1})$ with the direct integral $\int_{\mathcal{G}^0}^{\oplus} L^2(\lambda_u) d\mu(u)$ where, for each u , λ_u is the image of λ^u under inversion. Indeed, the closed support of each λ_u is $\mathcal{G}_u = \{x \mid d(x) = u\}$. Consequently Fubini's theorem implies that if $f \in L^2(\nu)$, then the field $\{f_u\}_{u \in \mathcal{G}^0}$ defined by the formula $f_u = f|_{\mathcal{G}_u}$ is measurable and in $\int_{\mathcal{G}^0}^{\oplus} L^2(\lambda_u) d\mu(u)$. And conversely, given $\{f_u\}_{u \in \mathcal{G}^0}$ in $\int_{\mathcal{G}^0}^{\oplus} L^2(\lambda_u) d\mu(u)$, f , defined by $f(x) = f_{d(x)}(x)$, belongs to $L^2(\nu^{-1})$. Observe too that for each $u \in \mathcal{G}^0$, $L^2(\lambda_u)$ is the Hilbert space carrying $\text{Ind } \delta_u$ where δ_u is the point mass at u .

PROPOSITION. *With this identification of $L^2(\nu^{-1})$ with $\int_{\mathcal{G}^0}^{\oplus} L^2(\lambda_u) d\mu(u)$, $\text{Ind } \mu$ may be identified with $\int_{\mathcal{G}^0}^{\oplus} \text{Ind } \delta_u d\mu$.*

PROOF. It suffices to show that $\text{Ind } \mu$ regarded as acting on $\int_{\mathcal{G}^0}^{\oplus} L^2(\lambda_u) d\mu$ commutes with all diagonal operators. But these, viewed as acting on $L^2(\nu^{-1})$, take the form $M(\varphi)$ where φ is a bounded measurable function on \mathcal{G}^0 and $M(\varphi)$ operates according to the formula

$$M(\varphi)\xi(x) = \varphi \circ d(x)\xi(x).$$

Now calculate: For $f \in C_c(\mathcal{G})$, $\varphi \in L^\infty(\mu)$, and $\xi \in L^2(\nu^{-1})$, we have on the one hand

$$(a) \quad M(\varphi)\text{Ind } \mu(f)\xi(x) = \int_{\mathcal{G}} f(xy)\varphi \circ d(x)\xi(y^{-1})\lambda^{d(x)}(y)$$

by 2.12, while on the other,

$$(b) \quad \text{Ind } \mu(f)M(\varphi)\xi(x) = \int_{\mathcal{G}} f(xy)\varphi \circ d(y^{-1})\xi(y^{-1})\lambda^{d(x)}(y).$$

Both integrals extend over $\{y \mid r(y) = d(x)\}$, but since on this set $d(y^{-1}) = r(y) = d(x)$, we conclude that the right-hand sides of (a) and (b) are the same. Thus $\text{Ind } \mu(f)$ commutes with the diagonal operators as asserted, and the proof is complete.

2.14 Let \mathcal{G} be a groupoid with left Haar system and let I be the intersection of the kernels of $\text{Ind } \mu$ as μ ranges over the measures of \mathcal{G}^0 . The quotient $C^*(\mathcal{G})/I$ is called the *reduced C*-algebra* of \mathcal{G} and is denoted by $C_{\text{red}}^*(\mathcal{G})$. This is not quite the definition given in [25, Definition 2.8, Chapter II], but it is easy to see that the two notions are equivalent. Observe that if \mathcal{G} is a group, then $\mathcal{G}^0 = \{e\}$, every measure on \mathcal{G}^0 is a multiple of the point mass at e , and the induced representation determined by such a measure is nothing but the regular representation of \mathcal{G} . Hence, in this case, I is the kernel of the regular representation and $C_{\text{red}}^*(\mathcal{G})$ has its usual meaning. In the group theoretic context, the vanishing of I is equivalent to the amenability of \mathcal{G} . One can formulate various notions of amenability for groupoids, but the relations between them are not clear and the subject is still fragmented (cf. [25, Chapter II, §3]).

2.15 PROPOSITION. *If \mathbb{G} is the groupoid obtained by reducing a transformation group groupoid $Y \times G$ to a locally closed subset of Y satisfying the conditions of 2.6.4 and if G is amenable, then $C^*(\mathbb{G}) = C_{\text{red}}^*(\mathbb{G})$.*

PROOF. This follows from Propositions 3.2, 3.7, and 3.9 in Chapter II of [25].

2.16 In the theory of transformation group C^* -algebras one may associate ideals with open invariant sets of the unit space [15]. The same holds for groupoid C^* -algebras. Let \mathbb{G} be a groupoid with Haar system λ and let U be an open invariant set in \mathbb{G}^0 . If $I_U = \{f \in C_c(\mathbb{G}) \mid f(x) = 0, x \notin \mathbb{G} \mid U\}$, then I_U is a two-sided ideal in $C_c(\mathbb{G})$ and its closure, $\overline{I_U}$, in $C_{\text{red}}^*(\mathbb{G}, \lambda)$ is a two-sided ideal in $C_{\text{red}}^*(\mathbb{G}, \lambda)$.

PROPOSITION [25, Chapter II, Proposition 4.4]. *The map $U \rightarrow \overline{I_U}$ is a one-to-one, order preserving map from the lattice of open invariant subsets of \mathbb{G}^0 into the lattice of two-sided ideals in $C_{\text{red}}^*(\mathbb{G}, \lambda)$. For each such set U , $\overline{I_U}$ is canonically isomorphic to $C_{\text{red}}^*(\mathbb{G} \mid U, 1_{\mathbb{G} \mid U} \lambda)$ and the quotient $C_{\text{red}}^*(\mathbb{G}, \lambda) / \overline{I_U}$ is canonically isomorphic to $C_{\text{red}}^*(\mathbb{G} \mid F, 1_{\mathbb{G} \mid F} \lambda)$ where $F = \mathbb{G}^0 \setminus U$.*

2.17 We will need to know when certain induced representations are faithful. The following proposition is the desideratum. The proof is modeled on Takai's proof of Proposition 2.1 in [27].

PROPOSITION. *Let \mathbb{G} be a groupoid with Haar system $\{\lambda^u \mid u \in \mathbb{G}^0\}$ and let μ be a measure on \mathbb{G}^0 . Then the kernel of $\text{Ind } \mu$ in $C_{\text{red}}^*(\mathbb{G}, \lambda)$ is the ideal $C_{\text{red}}^*(\mathbb{G} \mid U, 1_U \lambda)$ where U is the largest open invariant subset of \mathbb{G}^0 such that $\mu(U) = 0$.*

PROOF. It is evident from the definition of $\text{Ind } \mu$ and 2.16 that $C_{\text{red}}^*(\mathbb{G} \mid U, 1_U \lambda)$ is contained in the kernel of $\text{Ind } \mu$. Consequently, we may pass to the quotient $C_{\text{red}}^*(\mathbb{G} \mid F, 1_F \lambda)$, $F = \mathbb{G}^0 \setminus U$, and prove that $\text{Ind } \mu$ is faithful there. So assume, without loss of generality, that the smallest closed invariant set containing the support of μ is \mathbb{G}^0 itself. By Proposition 2.13, $\text{Ind } \mu$ is weakly contained in $\{\text{Ind } \delta_u \mid u \in \text{supp } \mu\}$ and of course each $\text{Ind } \delta_u$, $u \in \text{supp } \mu$, is weakly contained in $\text{Ind } \mu$, i.e., $\{\text{Ind } \mu\}$ and $\{\text{Ind } \delta_u \mid u \in \text{supp } \mu\}$ are weakly equivalent [11, 3.4.5]. Next observe that if u and v lie in the same orbit, then $\text{Ind } \delta_u$ and $\text{Ind } \delta_v$ are unitarily equivalent. Indeed by hypothesis, there is a $z \in \mathbb{G}$ such that $v = uz$. Consequently by the (right) invariance of $\{\lambda_u \mid u \in \mathbb{G}^0\}$ the map U , defined first for f in $C_c(\mathbb{G})$ by the formula

$$Uf(x) = \begin{cases} f(xz), & d(x) = u, \\ 0, & d(x) \neq u, \end{cases}$$

extends to a Hilbert space isomorphism from $L^2(\lambda_u)$ onto $L^2(\lambda_v)$ which, as an easy calculation shows, intertwines $\text{Ind } \delta_u$ and $\text{Ind } \delta_v$. Thus we find that $\{\text{Ind } \mu\}$ is weakly equivalent to $\{\text{Ind } \delta_u \mid u \text{ is in the orbit of an element in } \text{supp } \mu\}$. Since, however, the map from $C_c(\mathbb{G})$ to $C_c(\mathbb{G}^0)$ defined by the formula $f \rightarrow \int f(x) d\lambda_u(x)$ is continuous (2.4), it follows that for any set $S \subseteq \mathbb{G}^0$, $\{\text{Ind } \delta_u \mid u \in S\}$ is weakly equivalent to $\{\text{Ind } \delta_u \mid u \in \overline{S}\}$. Hence, by hypothesis, $\{\text{Ind } \mu\}$ is weakly equivalent to $\{\text{Ind } \delta_u \mid u \in \mathbb{G}^0\}$. Since the latter set contains $\text{Ind } \nu$ weakly for every measure ν , we conclude, by definition of $C_{\text{red}}^*(\mathbb{G}, \lambda)$ (2.14), that μ is faithful, as was to be shown.

2.17.1 Suppose that \mathbb{G} is the transitive groupoid based on a locally compact space X and that the Haar system $\{\lambda^u\}$ is $\{\delta_u \times \alpha\}_{u \in X}$ where $\text{supp } \alpha = X$. If $u \in X$, then since there is only one orbit, Proposition 2.17 implies that $\text{Ind } \delta_u$ is faithful on $C_{\text{red}}^*(\mathbb{G}, \lambda)$ for any $u \in X$. Of course one knows abstractly that in this case $C_{\text{red}}^*(\mathbb{G}, \lambda) = C^*(\mathbb{G}, \lambda)$ which, in turn, is isomorphic to the compact operators (cf. [25, Chapter II]) so that $\text{Ind } \delta_u$ is faithful. But let's verify part of this by constructing $\text{Ind } \delta_u$ explicitly. The measure ν induced by δ_u is simply $\delta_u \times \alpha$ and so ν^{-1} satisfies the equation $\int f d\nu^{-1} = \int f(x, u) d\alpha(x)$, $f \in C_c(\mathbb{G})$. Thus $L^2(\nu^{-1})$ may be viewed as $L^2(\alpha)$. But also,

$$\text{Ind } \delta_u(f)\xi(x, y) = \int_{\mathbb{G}} f((x, y)(x_1, y_1))\xi(y_1, x_1) d(\delta_u \times \alpha)(x_1, y_1).$$

This forces $y = x_1 = u$, and so the integral is

$$\int_X f(x, y_1)\xi(y_1, u) d\alpha(y_1)$$

which makes it perfectly clear that the image of $\text{Ind } \delta_u$ is the algebra of compact operators on $L^2(\nu^{-1})$.

2.18 EXAMPLE 2.3 CONCLUDED. Since \mathbf{R} is commutative, and therefore amenable, we may apply 2.15 to conclude that $C^*(\mathbb{G} | X) = C_{\text{red}}^*(\mathbb{G} | X)$. We know that $[0, \infty)$ is an open invariant set in the unit space X of $\mathbb{G} | X$ and so $C^*((\mathbb{G} | X) | [0, \infty)) = C^*(\mathbb{G} | [0, \infty))$ may be viewed as an ideal in $C^*(\mathbb{G} | X)$ by Proposition 2.16. But we showed in 2.3 that $\mathbb{G} | [0, \infty)$ is topologically isomorphic to the transitive groupoid on $[0, \infty)$ so that by 2.7.1, $C^*(\mathbb{G} | [0, \infty))$ is isomorphic to \mathcal{K} , the compact operators. By Proposition 2.16 we conclude that $C^*(\mathbb{G} | X)/\mathcal{K} \cong C^*(\mathbb{G} | \{\infty\}) \cong C_0(\mathbf{R})$ since $\mathbb{G} | \{\infty\}$ is topologically isomorphic to \mathbf{R} and the group C*-algebra of \mathbf{R} is $C_0(\mathbf{R})$. The map U from $L^2([0, \infty))$ to $L^2(\lambda^0)$ defined by the formula

$$U\xi(x, t) = \begin{cases} \xi(t), & x = 0, \\ 0, & x \neq 0, \end{cases}$$

is a Hilbert space isomorphism and the computations of 2.12.2 show that for h in $C_c(\mathbb{G} | X)$,

$$(U^{-1}J^{-1}\text{Ind } \delta_0(h)JU)\xi(t) = \int_{-\infty}^{\infty} h(t, s)1_{[0, \infty)}(t+s)\xi(t+s) ds$$

for all $\xi \in L^2([0, \infty))$. This is equation (1.3). Since the orbit of 0 is dense in X , Proposition 2.17 implies that $\text{Ind } \delta_0$ is faithful. The image of $U^{-1}J^{-1}\text{Ind } \delta_0JU$ contains \mathfrak{B} , the C*-algebra generated by the Wiener-Hopf operators on $L^2([0, \infty))$. So to complete the derivation of the properties of \mathfrak{B} from the groupoid perspective, all that remains is to show that $C_c(\mathbb{G} | X)$ is generated by the functions of the form $f(x, t) = \tilde{f}(t)$ where \tilde{f} is a function in $C_c(\mathbf{R})$. This will be done in the next section, Proposition 3.5.

3. Wiener-Hopf operators on semigroups. In this section we present our basic construction and put into evidence the parameters needed to describe the C*-algebra of Wiener-Hopf operators in the most general possible setting.

3.1 Throughout, G will denote a second countable, locally compact group with identity e and left Haar measure λ fixed once and for all. Right Haar measure will be denoted by λ^{-1} . Also, P will denote a closed subset of G which is the closure of its interior, $\text{Int } P$, such that

- (i) $PP \subseteq P$;
- (ii) $e \in P \cap P^{-1}$;
- (iii) P generates G ; and
- (iv) $tPt^{-1} \subseteq P$ for all $t \in G$.

Thus P is a closed normal subsemigroup of positive measure containing the identity of G .

Observe that $\text{Int } P$ satisfies (i), (iii), and (iv) also. Consequently, when we use $\text{Int } P$ to define a relation $<_P$ on G by the prescription, $a <_P b$ if and only if $ba^{-1} \in \text{Int } P$, we see that $<_P$ is transitive by (i) and that $<_P$ directs G by virtue of (iii) and (iv) (see the proof of Proposition 1 in Chapter II of [17]). Condition (iv) means that left and right multiplication are monotone operations. Of course P itself defines a similar relation which, in fact, is a preorder in the sense that two elements may dominate each other without being equal; it is a partial order precisely when $P \cap P^{-1} = \{e\}$. We shall denote this order by \leq_P .

Note that if G is connected, then condition (iii) is redundant. The reason is that the subgroup generated by P is open, and in every locally compact group an open subgroup is closed. Our basic construction involves only the semigroup structure of P spelled out in (i)–(iv), the orderings $<_P$ and \leq_P won't explicitly enter the discussion for a little while (3.9ff). We note in passing that when we investigate our examples, it will be convenient to allow $P \cap P^{-1}$ to be more than just $\{e\}$ (cf. 3.12).

3.2 We write A for the C^* -algebra of bounded continuous functions on G generated by the functions of the form $1_P * f$ where $f \in L^1(\lambda^{-1})$ and where, by definition, $1_P * f(t) = \int 1_P(ts)f(s^{-1})d\lambda(s) = \int_{P^{-1}t} f(s)d\lambda^{-1}(s)$. The maximal ideal space of A will be denoted by Y , and for $\varphi \in A$ and $y \in Y$, we write $\hat{\varphi}(y)$ for the value of the Gelfand transform of φ at y . Observe that the norm of $1_P * f$ in A is bounded by the norm of f in $L^1(\lambda^{-1})$ and that if f moves continuously through $L^1(\lambda^{-1})$, then $1_P * f$ moves continuously through A . It follows that the functions in A are, in fact, uniformly continuous on G . Consequently, if for $y \in Y$ and $t \in G$ we define yt in Y by the formula $(1_P * f)^\wedge(yt) = (1_P * f_t)^\wedge(y)$, $f \in L^1(\lambda^{-1})$, where $f_t(s) = f(st^{-1})$, we find that yt is well defined and that the map from $Y \times G$ to Y defined by the formula $(y, t) \rightarrow yt$ is continuous, converting (Y, G) into a locally compact transformation group. Since evaluation of a function in A at e defines an element y_e in Y , we obtain a continuous mapping τ from G into Y defined by the formula $\tau(s) = y_e s$. We let X be the closure of $\tau(P)$.

3.3 LEMMA. (i) *The range of τ is dense in Y .*

(ii) *The isotropy group of y_e is $P \cap P^{-1}$; so, in particular, τ is one-to-one if and only if $P \cap P^{-1} = \{e\}$.*

(iii) *The space X is compact.*

(iv) *Each orbit in Y meets X .*

(v) *For each $x \in X$, the set $\mathfrak{T}_x = \{t \mid xt \in X\}$ is a closed subset of G containing P such that $\mathfrak{T}_x P = P \mathfrak{T}_x \subseteq \mathfrak{T}_x$.*

PROOF. (i) If φ is a function in $C_0(Y)$ which vanishes on $\tau(G)$, then by definition of τ , φ is the Gelfand transform of the zero function in A . Thus $\varphi \equiv 0$, and $\tau(G)$ is dense.

(ii) Let H be the isotropy group of y_e . To see that $H = P \cap P^{-1}$, simply note that $y_e t = y_e$ if and only if $1_P * f(t) = 1_P * f(e)$ for all f in $L^1(\lambda^{-1})$, equivalently, if and only if $1_{P^{-1}} * f(e) = 1_P * f(e)$ for all f in $L^1(\lambda^{-1})$. It follows that t lies in H if and only if $1_{P^{-1}} = 1_P$ a.e. But P is closed and λ is regular, so we conclude that t is in H if and only if $Pt^{-1} = P$ which is tantamount to the assertion that t belongs to $P \cap P^{-1}$. Thus $H = P \cap P^{-1}$.

(iii) To see that X is compact, simply observe that A contains a function which is identically 1 on P . Indeed, $1_P * f$, where f is any function in $L^1(\lambda^{-1})$ supported on P^{-1} such that $\int f d\lambda^{-1} = 1$, will do.

(iv) Fix y in Y and choose a sequence $\{t_n\}_{n=1}^\infty$ in G such that $\tau(t_n) = y_e t_n$ converges to y . If, for some t in G , infinitely many of the points $y_e t_n$ lie in Xt , then by (iii) yt^{-1} lies in X , i.e., X meets the orbit of y . So suppose that for each t only finitely many points $y_e t_n$ lie in Xt . Then for each t , at most finitely many t_n lie in Pt . This means that the functions $1_{P^{-1}t_n}$ converge to zero pointwise. Hence, by the dominated convergence theorem, we conclude that for every f in $L^1(\lambda^{-1})$,

$$\begin{aligned} (1_P * f)^\wedge(y) &= \lim (1_P * f)^\wedge(y_e t_n) = \lim (1_P * f)(t_n) \\ &= \lim \int 1_{P^{-1}t_n}(s) f(s) d\lambda^{-1}(s) = 0. \end{aligned}$$

Thus y represents the zero multiplicative linear functional on A which, by convention, is not included in the maximal ideal space of any Banach algebra. Thus for some t , infinitely many of the points $y_e t_n$ lie in Xt and we conclude that yt belongs to X .

(v) Evidently, \mathfrak{T}_x is closed and contains P . If $t \in \mathfrak{T}_x$ and if $s \in P$, then $x(ts) = (xt)s$ lies in X while $x(st) = xt(t^{-1}st)$ also lies in X , and this proves (v).

We remark in passing that in all examples we know of, \mathfrak{T}_x is actually a semigroup, but we are unable to prove this in general.

3.4 Form the groupoid $Y \times G$ as described in the preceding section (see 2.2.4) and let the Haar system on $Y \times G$ be $\{\delta_y \times \lambda\}_{y \in Y}$ as in 2.5. The groupoid in which we are interested is the reduction of $Y \times G$ to X and we shall denote it by \mathfrak{G} . Thus $\mathfrak{G} = \{(x, s) \in Y \times G \mid x \in X \text{ and } xs \in X\}$. Observe that for $x \in X$, $(x, e) \in \mathfrak{G}^0$, and $r^{-1}(x, e) = \{(x, t) \mid t \in \mathfrak{T}_x\}$. Note, too, that since P is the closure of its interior, Lemma 3.3(v) implies that \mathfrak{T}_x is the closure of its interior. Consequently, we find that for each $x \in X$, the support of $1_{\mathfrak{G}}(\delta_x \times \lambda)$ is all of $r^{-1}(x, e) \cap \mathfrak{G}$. Thus $\{1_{\mathfrak{G}}\delta_x \times \lambda\}_{x \in X}$ forms a Haar system for \mathfrak{G} (2.5), and this is the one we shall choose for the remainder of the paper.

For f in $C_c(G)$, we define \tilde{f} in $C_c(\mathfrak{G})$ by the formula $\tilde{f}(x, s) = f(s)$.

3.5 PROPOSITION. *The subalgebra generated by $\{\tilde{f} \mid f \in C_c(G)\}$ is dense in $C_c(\mathfrak{G})$ with the inductive limit topology and therefore it is dense in $C^*(\mathfrak{G})$.*

PROOF. First note that if f is in $C_c(G)$, then for y in Y , $\int_G 1_X(yt)f(t^{-1})d\lambda(t) = (1_P * \hat{f})(y)$ by definition of X . Since $C_c(G)$ is dense in $L^1(\lambda^{-1})$, it follows that the algebra generated by the functions of the form $\psi(y) = \int 1_X(yt)f(t)d\lambda(t)$ is dense in $C_0(Y)$. (Note that since $C_c(G)$ is invariant under inversion, it is alright to define ψ as we do, rather than as $\int 1_X(yt)f(t^{-1})d\lambda(t)$.) Thus to show that the algebra generated by $\{\tilde{f}|f \in C_c(G)\}$ is dense in $C_c(\mathfrak{G})$, it suffices to prove that a function φ in $C_c(\mathfrak{G})$ of the form

$$(*) \quad \varphi(x, s) = \left(\int 1_X(xt)f(t)d\lambda(t) \right) g(s)$$

where f and g are in $C_c(G)$ and $(x, t) \in \mathfrak{G}$ can be approximated by elements in this subalgebra.

To this end, fix f and $g \in C_c(G)$. Let K be a compact neighborhood of the support of g , $\text{supp}(g)$, let $\varepsilon > 0$ be given, and let $M = \int |f(t)|d\lambda(t)$. Next choose an open neighborhood V of e in G such that if $sr^{-1} \in V$, then $|g(s) - g(r)| \leq \varepsilon/M$, and such that the product $V\text{supp}(g)$ is contained in K . Since the support of f is compact, we may find $t_1, \dots, t_n \in G$ such that $\{Vt_i\}_{i=1}^n$ covers $\text{supp}(f)$. Construct a partition of unity $\{h_i\}_{i=1}^n$ on $\text{supp}(f)$ subordinate to this cover. Thus, $\text{supp}(h_i) \subseteq Vt_i$, $i = 1, \dots, n$, and $\sum_{i=1}^n h_i(r) = 1$, $r \in \text{supp}(f)$. Set $f_i(r) = f(r)h_i(r)$, and set $g_i(r) = g(t_i r)$, $r \in G$, $i = 1, \dots, n$. We assert that the function $\psi(x, s) = \sum_{i=1}^n \tilde{f}_i * \tilde{g}_i(x, s)$, which lies in the algebra generated by $\{\tilde{f}|f \in C_c(G)\}$, provides the desired approximation to φ , the function defined by equation (*). Utilizing the formula in paragraph 2.6.4, we find that ψ is given by the formula

$$\begin{aligned} \psi(x, s) &= \sum_{i=1}^n \int 1_X(xt)\tilde{f}_i(x, t)\tilde{g}_i(xt, t^{-1}s)d\lambda(t) \\ &= \sum_{i=1}^n \int 1_X(xt)f_i(t)g(t_it^{-1}s)d\lambda(t). \end{aligned}$$

Consider each term separately. Since f_i is supported in Vt_i , the i th integrand vanishes outside of Vt_i . Observe that $t_it^{-1}s$ is in the support of g if and only if s belongs to $t_i^{-1}\text{supp}(g)$. Consequently in the i th integral where t is constrained to Vt_i , we see that when $t_it^{-1}s$ lies in $\text{supp}(g)$, s belongs to $V\text{supp}(g)$ which, by assumption, belongs to K . Thus each term of the sum defining ψ , and therefore ψ itself, is supported in $X \times K$. That is, φ and ψ have common compact support. Calculate to find that

$$\begin{aligned} \varphi(x, s) - \psi(x, s) &= \left(\int 1_X(xt)f(t)d\lambda(t) \right) g(s) \\ &\quad - \sum_{i=1}^n \int 1_X(xt)f_i(t)g_i(t^{-1}s)d\lambda(t) \\ &= \sum_{i=1}^n \int 1_X(xt)f_i(t)(g(s) - g(t_it^{-1}s))d\lambda(t). \end{aligned}$$

Consequently,

$$|\varphi(x, s) - \psi(x, s)| \leq \sum_{i=1}^n \int_G |f_i(t)| |g(s) - g(t_i t^{-1} s)| d\lambda(t).$$

But in the i th integral of this sum, the integrand is supported on Vt_i , and for $t \in Vt_i$, $s(t_i t^{-1} s)^{-1} = t t_i^{-1} \in V$. Thus the integrand in the i th integral is dominated by $|f_i(t)| \varepsilon / M$ and so the sum is dominated by $(\varepsilon / M) \sum_{i=1}^n \int |f_i(t)| d\lambda(t) = (\varepsilon / M) \int |f(t)| d\lambda(t) = \varepsilon$. This shows that $|\varphi(x, s) - \psi(x, s)| < \varepsilon$ on $X \times K$ and completes the proof.

3.6 For f in $C_c(G)$, we define the *Wiener-Hopf operator* $W(f)$ on $L^2(P, \lambda)$ by the formula

$$W(f)\xi(t) = \int_G f(s)\xi(ts)1_P(ts) d\lambda(s),$$

$\xi \in L^2(P, \lambda)$. It is easy to see that $W(f)$ is a bounded linear operator on $L^2(P, \lambda)$ with norm dominated by the $L^1(\lambda^{-1})$ -norm of f . Thus, in particular, the map $f \rightarrow W(f)$ is continuous with respect to the inductive limit topology on $C_c(G)$ and the norm topology on $\mathfrak{L}(L^2(P, \lambda))$. Observe that $W(f)$ is an immediate generalization of the operator in formula (1.2). The C^* -algebra generated by $\{W(f) | f \in C_c(G)\}$ will be denoted by \mathfrak{B} or $\mathfrak{B}(P)$ and will be referred to as the *C^* -algebra of Wiener-Hopf operators on P* .

We extend W to a representation of $C_c(\mathfrak{G})$ which, keeping the same name W , is defined by the formula

$$W(f)\xi(t) = \int f(y_e t, s)\xi(ts)1_P(ts) d\lambda(s),$$

$f \in C_c(\mathfrak{G})$, $\xi \in L^2(P, \lambda)$. It is a straightforward calculation, which we omit, to show that W is indeed a representation of $C_c(\mathfrak{G})$ that is contractive in the $L^1(\mathfrak{G})$ -norm. Thus W extends to a representation of $C^*(\mathfrak{G})$ on $L^2(P, \lambda)$. By virtue of the continuity properties of W on $C_c(G)$, we may conclude from Proposition 3.5 that the image of $C^*(\mathfrak{G})$ under W is \mathfrak{B} .

Let U map $L^2(P, \lambda)$ to $L^2(\mathfrak{G}, \lambda^{y_e})$ according to the formula

$$U\xi(x, t) = \begin{cases} \xi(t), & x = y_e, \\ 0, & \text{otherwise,} \end{cases}$$

and observe that U is indeed a Hilbert space isomorphism with inverse defined by the formula $(U^{-1}\xi)(t) = \xi(y_e, t)$. Now calculate to find that for $f \in C_c(\mathfrak{G})$ and $\xi \in L^2(\lambda^{y_e})$, we have

$$UW(f)U^{-1}\xi(y_e, t) = \int f(y_e t, s)\xi(y_e, ts)1_X(y_e ts) d\lambda(s).$$

But in view of the calculations in 2.12.2, we see that $W(f)$ is unitarily equivalent to $\text{Ind } \delta_{y_e}(f)$ and that the unitary equivalence is implemented by JU where J is the Hilbert space isomorphism between $L^2(\lambda^{y_e})$ and $L^2(\lambda_{y_e})$ defined by the equation $(J\xi)(x, t) = \xi(xt, t^{-1})$.

In view of the formal similarities between the Wiener-Hopf algebra $\mathfrak{B}(P)$ and the regular representation of a group, the following theorem might well be anticipated. However, its proof, in its entirety, appears more involved than one would expect.

3.7 THEOREM. *The representation W is faithful on $C_{\text{red}}^*(\mathfrak{G})$ with range equal to all of $\mathfrak{B}(P)$.*

PROOF. All that needs to be shown is that W is faithful. Since W is equivalent to $\text{Ind } \delta_{y_e}$, we need only show that this representation is faithful. By Proposition 2.17 the kernel of $\text{Ind } \delta_{y_e}$ is $C_{\text{red}}^*(\mathfrak{G} \mid U)$ where U is the largest open invariant set of X not containing y_e , but by Lemma 3.3, U is empty. Thus W is faithful.

3.7.1 Suppose that the group $H = P \cap P^{-1}$ is a factor of G so that there is a closed subgroup K of G such that G is isomorphic to $K \times H$ under the map $(k, h) \rightarrow kh$. We may of course identify K with G/H . In so doing, we write \bar{P} for the closure of the image of P in K under the quotient map. Letting K play the role of G and \bar{P} the role of P , we obtain a new groupoid $\bar{\mathfrak{G}}$ with the obvious left Haar system. A left Haar system for $\bar{\mathfrak{G}} \times H$, then, is simply the product of the left Haar system for $\bar{\mathfrak{G}}$ and left Haar measure on H . Assuming all the Haar measures involved are normalized so that Haar measure on G is the product of Haar measure on K with that on H , we may conclude that \mathfrak{G} and $\bar{\mathfrak{G}} \times H$ are canonically, topologically isomorphic. It results, therefore, that $C_{\text{red}}^*(\mathfrak{G})$ is isomorphic to $C_{\text{red}}^*(\bar{\mathfrak{G}}) \otimes C_{\text{red}}^*(H)$ where the norm on the tensor product is the least C^* -cross norm. (This is proved in Proposition 6 in §4 of [21] for groups, but it is evident that the arguments there work for groupoids as well.) We thus arrive at the following corollary to Theorem 3.7 which generalizes a result of Gol'denšteĭn and Gohberg [19].

COROLLARY. *With the notation just established we find that $\mathfrak{B}(P)$ is isomorphic to $\mathfrak{B}(\bar{P}) \otimes C_{\text{red}}^*(H)$ where the tensor product is endowed with the least C^* -cross norm.*

3.7.2 If $P \cap P^{-1} = \{e\}$, then by Lemma 3.3, τ is one-to-one and X may be viewed as a compactification of P . Recall that in general a compactification of a topological space E is a pair (ι, F) where F is compact and ι is a continuous one-to-one map of E onto a dense subset of F . The compactification is called *regular* if $\iota(E)$ is open in F and ι is a homeomorphism onto its range.

COROLLARY. *If $P \cap P^{-1} = \{e\}$ and X is a regular compactification of P , then \mathfrak{B} contains the ideal \mathfrak{K} of compact operators on $L^2(P, \lambda)$.*

PROOF. By hypothesis $\tau(P)$ is an open invariant subset of X and the groupoid $\mathfrak{G} \mid \tau(P)$ is transitive and principal. Indeed, $\mathfrak{G} \mid \tau(P)$ is topologically isomorphic to the trivial groupoid on P under the map $(\tau(t), s) \rightarrow (ts, t)$ (cf. Example 2.3). By 2.7.1, $C^*(\mathfrak{G} \mid \tau(P))$ is isomorphic to the compact operators. Since W is unitarily equivalent to $\text{Ind } \delta_e$, it is clear that W maps $C^*(\mathfrak{G} \mid \tau(P))$ onto \mathfrak{K} .

3.7.3 It is attractive to conjecture that the converse of Corollary 3.7.2 is true. However, we are unable to prove it.

By way of example, we note that X need not be a regular compactification of P . Indeed, Douglas [12] has shown that if G is a discrete subgroup of \mathbf{R} which is not singly generated and if P is the semigroup of nonnegative elements in G , then $\mathfrak{B}(P)$ does not contain any nonzero compact operator. By Corollary 3.7.2, then, X is not a regular compactification of P .

In general, it seems quite difficult to decide if X is a regular compactification of P , particularly when the group G is discrete. Also, this related “obvious” question seems difficult to answer: If X is a regular compactification of P , must $\mathfrak{B}(P)$ be type I?

3.8 To develop a better understanding of \mathfrak{B} and, in particular, to utilize the theory of groupoid C*-algebras in the analysis of \mathfrak{B} , we must have a clearer picture of X . As a compactification of P (assuming for a moment that $P \cap P^{-1} = \{e\}$), X labels all the various modes of approach to infinity along P .³ To make this vague statement precise and to put into evidence the key idea of our analysis, we consider the special, but revealing case where $G = \mathbf{R}^2$ and $P = [0, \infty) \times [0, \infty)$. Let $f \in L^1(\mathbf{R}^2)$ and set $\varphi = 1_P * f$. If $\{(x_n, y_n)\}_{n=1}^\infty$ and $\{(x'_n, y'_n)\}_{n=1}^\infty$ are two sequences which go to infinity while remaining *interior* to P , then it is easy to see that $\lim \varphi(x_n, y_n) = \lim \varphi(x'_n, y'_n)$. On the other hand, if we fix x , take the limit $\lim_{y \rightarrow \infty} \varphi(x, y)$, and then let x vary, we obtain a continuous function on $[0, \infty) \times \{0\}$ which has a limit as $x \rightarrow \infty$. Likewise, if we fix y , take the limit $\lim_{x \rightarrow \infty} \varphi(x, y)$, and then let y vary, we obtain a continuous function on $\{0\} \times [0, \infty)$ having a limit as $y \rightarrow \infty$. Thus, it seems reasonable that as a set we might be able to identify X with

$$P \cup [0, \infty) \times \{0\} \cup \{0\} \times [0, \infty) \cup \{\infty\}$$

where the union is a *disjoint* union and where we wish to distinguish between the roles of the two faces $[0, \infty) \times \{0\}$ and $\{0\} \times [0, \infty)$ as subsets of P and as separate entities in the union. As we shall see in the next section, this is indeed the case. That is, there *is* a topology on the indicated union such that the resulting topological space is homeomorphic to X . This example, then, points to the desirability of finding the facial structure of P and suggests a natural candidate for a model for X . The remainder of the section is devoted to building this model.

3.9 A *face* of P is by definition a closed subsemigroup F of P satisfying the following conditions.

- (i) $e \in F$;
- (ii) $tFt^{-1} \subseteq F$ for all t in G ;
- (iii) if $x \in P$, $y \in F$, and if $x \leq_P y$, then $x \in F$; and
- (iv) if we write $\langle F \rangle$ for the closed subgroup generated by F , and if $\text{Int}(F)$ denotes the interior of F with respect to the relative topology on $\langle F \rangle$, then F is the closure of $\text{Int}(F)$.

We want to emphasize that we count $\{e\}$ and P among the faces of P .

³Simonenko [26] also introduced a compactification of P ; but his is different from ours and serves a different purpose.

- 3.10 LEMMA. (i) $\text{Int}(F)$ is an open subsemigroup of $\langle F \rangle$ which generates $\langle F \rangle$.
(ii) $t \text{Int}(F) t^{-1} \subseteq \text{Int}(F)$ for all $t \in G$.
(iii) If, for elements $s, t \in \langle F \rangle$, the relation $t <_F s$ is defined by stipulating that $t^{-1}s \in \text{Int}(F)$, then $<_F$ is a directed, transitive relation on $\langle F \rangle$.
(iv) $\text{Int}(F)(\text{Int}(F))^{-1} = \langle F \rangle$.

PROOF. Assertions (i) and (ii) are evident. Assertion (iii) is just a restatement of the fact proved in paragraph 3.1 concerning the relation $<_P$ on G . Assertion (iv) follows easily from (i) and (ii).

3.11 Let f be a function on $\langle F \rangle$. In the sequel, we shall write $\lim_{t \in \langle F \rangle} f(t)$ or simply $\lim_t f(t)$ for the limit, if it exists, of the net $\{f(t)\}_{t \in \langle F \rangle}$ taken along the set $\langle F \rangle$ directed by $<_F$.

PROPOSITION. (i) For each $s \in G$, $\lim_{t \in \langle F \rangle} 1_{tP^{-1}}(s) = 1_{\langle F \rangle P^{-1}}(s)$.
(ii) If $f \in L^1(\lambda^{-1})$ and if $\varphi(t) = 1_P * f(t) = \int_{tP^{-1}} f(s) d\lambda^{-1}(s)$, then for $t \in G$, we have $\lim_{s \in \langle F \rangle} \varphi(ts) = \int_{t\langle F \rangle P^{-1}} f(u) du$ and the convergence is uniform in t as long as t is restricted to compact subsets of G .

PROOF. (i) The net $\{1_{tP^{-1}}\}_{t \in \langle F \rangle}$ is a uniformly bounded, increasing net of bounded functions and, therefore, it converges to its supremum. Since $\text{Int}(F)(\text{Int}(F))^{-1} = \langle F \rangle$ by Lemma 3.10, this supremum is $1_{\langle F \rangle P^{-1}}$.

(ii) The limiting assertion follows from (i) and the dominated convergence theorem. The uniformity assertion follows from the fact that translation is continuous on L^1 .

3.12 Observe that $\langle F \rangle P^{-1} = (\langle F \rangle P)^{-1}$ and that $\langle F \rangle P$ is a semigroup of the type we have been considering except that $\langle F \rangle P$ need not be closed. (Perhaps the simplest example is afforded by the forward light cone P in \mathbf{R}^3 : $P = \{(x_1, x_2, x_3) \mid x_3 \geq 0, x_3^2 \geq x_1^2 + x_2^2\}$. The only nontrivial faces of P are the rays on the surface of P . If F is such a ray, then it is geometrically obvious that $\langle F \rangle + P$ is the union of $\langle F \rangle$ and the open half-space containing $\text{Int}(P)$ determined by the tangent plane to P through F .) In any event, it does no harm to replace $\langle F \rangle P$ by its closure because the integrals we will be considering are the same when extended over one or the other semigroup.

Note that even if $P \cap P^{-1} = \{e\}$ the closure of $\langle F \rangle P$ fails to have this property unless $F = \{e\}$. This is the principal reason for not requiring the condition $P \cap P^{-1} = \{e\}$ earlier in our analysis of $\mathfrak{B}(P)$. Note, too, that in general $H(F)$, which is defined to be $(\text{closure } \langle F \rangle P) \cap (\text{closure } \langle F \rangle P)^{-1}$ is larger than $\langle F \rangle$. Indeed, in the example just given, $H(F)$ is the entire tangent plane through F while $\langle F \rangle$ is just a line. Finally, observe that $H(F)P$ is closed and that $H(F)P \cap (H(F)P)^{-1} = H(F)$.

3.13 Let \mathfrak{F} denote the faces of P and keep in mind that $\{e\}$ and P belong to \mathfrak{F} . If H is a (closed) normal subgroup and if $x \in G$, we write $[x]$ for the image of x under the quotient map of G onto G/H ; thus, $[x]$ is the coset of x in G/H . Our model \tilde{Y} for Y in the construction of the groupoid \mathfrak{G} is now easily described. By definition, $\tilde{Y} = \{(F, [x]) \mid F \in \mathfrak{F} \text{ and } [x] \in G/H(F)\}$. Observe that \tilde{Y} carries a natural action of G , namely the quotient action. That is, for $(F, [x]) \in \tilde{Y}$ and $t \in G$, $(F, [x])t$ is

defined to be $(F, [xt])$. It is immediate that two points $(F_1, [x_1])$ and $(F_2, [x_2])$ lie in the same orbit if and only if $F_1 = F_2$ and that the isotropy group of any point $(F, [x])$ is $H(F)$.

Suppose that $\varphi = 1_P * f$, $f \in L^1(\lambda^{-1})$. We may regard φ as a function $\tilde{\varphi}$ on $\{(\{e\}, [x]) \mid x \in G\} \subseteq \tilde{Y}$ simply by setting $\tilde{\varphi}(\{e\}, [x]) = \varphi(x)$. From this perspective, $\tilde{\varphi}$ has a natural extension to all of \tilde{Y} . Keeping the same name, $\tilde{\varphi}$ is given by the formula

$$\tilde{\varphi}(F, [x]) = \int_{xH(F)P^{-1}} f(t) d\lambda^{-1}(t).$$

For reference, note that since $H(F)P = \text{closure}(\langle F \rangle P)$, we may also write $\tilde{\varphi}(F, [x]) = \int_{x\langle F \rangle P^{-1}} f(t) d\lambda^{-1}(t)$. The point to keep in mind is that while experimentation with the quarter plane suggests that the model \tilde{Y} ought to be $\{(F, [x]) \mid [x] \in G/\langle F \rangle\}$, the function $\tilde{\varphi}$ won't necessarily separate the points of \tilde{Y} . To achieve this, it is necessary to divide out the frequently larger groups $H(F)$. We don't know, however, if this is sufficient in general.

The model \tilde{X} for X is taken to be the set of all $(F, [x])$ in Y such that $[x]$ lies in the closure of the image of P under the quotient map from G to $G/H(F)$. In our examples, we will find that \tilde{Y} carries a natural, locally compact topology, that \tilde{X} is compact, that the functions $\tilde{\varphi}$ are continuous on \tilde{Y} and separate the points of \tilde{Y} , that the action of G on \tilde{Y} is continuous, and that the map $\varphi \rightarrow \tilde{\varphi}$ extends to an equivariant isomorphism from A onto $C_0(\tilde{Y})$. It will follow, then, that \tilde{Y} and \tilde{X} are indeed the objects we are looking for. Since the orbits are put clearly into evidence, we will be able to apply the methods of [25] freely to dissect $\mathfrak{B}(P)$.

The topology on \tilde{Y} is actually suggested by the very description of \tilde{Y} . In the cases we consider, those in which P is a certain type of cone in \mathbf{R}^n , \mathfrak{F} will have a natural topology and \tilde{Y} will have the structure of a vector bundle over \mathfrak{F} . The space \mathfrak{F} won't be connected and the dimension of the fibers of \tilde{Y} will vary. It will develop that certain "components" of \tilde{Y} have natural, locally compact, Hausdorff topologies and that Proposition 3.11 may be used to glue these "components" together.

Lest the reader believe that our approach is a panacea, i.e., that X and Y may always be modeled in the way just described, from the facial structure of P , we note that when G is a countable discrete subgroup of \mathbf{R} , say, which is not singly generated, and when $P = G \cap [0, \infty)$, then $\{e\}$ and P are the only faces of P by virtue of the Archimedian order. But \tilde{Y} cannot be a model for Y if only for the reason that \tilde{Y} is countable and Y isn't.

4. Polyhedral cones. By definition, a *polyhedral cone* in \mathbf{R}^n is the closed cone with vertex 0 generated by a finite number of points. A good reference for the basic geometric facts about such cones is Klee's paper [23]. We assume that our cone P spans \mathbf{R}^n , but we don't assume that the points defining P are linearly independent; i.e., we don't assume that P is isomorphic to the positive orthant $\{(x_1, \dots, x_n) \mid x_i \geq 0\}$ in \mathbf{R}^n . Our objective in this section is to present a rather complete description of $\mathfrak{B}(P)$. In particular, we show that $\mathfrak{B}(P)$ is solvable and we identify explicitly the

consecutive quotients in a natural composition series for $\mathfrak{B}(P)$. Our results generalize those of Dynin who analyzed the positive orthant in [14] (cf. [13], also).

4.1 Throughout this section, P will denote a polyhedral cone in \mathbf{R}^n . Since in the remainder of the paper each of the cones considered is contained in some Euclidean space, we will use additive notation from now on. By virtue of Corollary 3.7.1, we may assume that $P \cap -P = \{0\}$; i.e., as one says, we may assume that P contains no line. This we shall do expect where explicitly stated otherwise. According to Corollary 2.3 of [23], then, P is generated by its extreme rays. We fix points x_1, \dots, x_N which generate the extreme rays of P (and we assume, of course, that no two of the x_i lie on the same ray). As in §3, we let \mathfrak{F} denote the collection of all faces of P . Since P is polyhedral, \mathfrak{F} is a finite set. Indeed, since each face of P is uniquely determined by a subset of $\{x_1, \dots, x_N\}$ it follows that there are at most 2^N faces in all. Recall that for $F \in \mathfrak{F}$, $\langle F \rangle$ denotes the subgroup $F - F$ generated by F , so that in the present setting $\langle F \rangle$ is, in fact, a linear subspace of \mathbf{R}^n . We write \mathfrak{F}_k for $\{F \in \mathfrak{F} \mid \dim \langle F \rangle = k\}$. In particular, then, $\mathfrak{F}_0 = \{0\}$ while $\mathfrak{F}_n = \{P\}$. The Euclidean structure at our disposal allows us to identify the quotient $\mathbf{R}^n / \langle F \rangle$ with the orthogonal complement $\langle F \rangle^\perp = F^\perp$ of F for each F in \mathfrak{F} . This we shall do, and we shall write $U(F)$ (resp. $U(F^\perp)$) for the orthogonal projection of \mathbf{R}^n onto $\langle F \rangle$ (resp. F^\perp). The reason that we adopt here what appears to be somewhat unorthodox notation stems from the desire to maintain a certain amount of uniformity throughout the paper. In the next two sections, the cones studied are related to certain Jordan algebras. The so-called quadratic representation of such an algebra enters into the parametrization of the faces of the cones, and we use the fairly standard notion U for this representation.

4.2 LEMMA. *Let F be a face of P . Then*

- (i) $\langle F \rangle + P$ is closed;
- (ii) $\langle F \rangle + P = \langle F \rangle \oplus F_+^\perp$, where $F_+^\perp = U(F^\perp)P$; and
- (iii) $H(F) = \langle F \rangle$, where $H(F) = (\langle F \rangle + P) \cap -(\langle F \rangle + P)$ (cf. 3.12).

PROOF. (i) Suppose, without loss of generality, that x_1, \dots, x_k generate F . Since $\langle F \rangle = F - F$, $\langle F \rangle + P$ is the cone generated by $x_1, \dots, x_N, -x_1, \dots, -x_k$. But since it is the image under a linear map of the positive orthant in \mathbf{R}^{N+k} , it is closed by [23, Corollary 2.5].

(ii) If $x + z \in \langle F \rangle + P$ with $x \in \langle F \rangle$ and $z \in P$, then $x + z = (x + U(F)z) + U(F^\perp)z$ belongs to $\langle F \rangle \oplus F_+^\perp$. On the other hand, if $x + y \in \langle F \rangle \oplus F_+^\perp$, we may choose $z \in P$ so that $y = U(F^\perp)z$. If $x' = U(F)z$, then $x + y = (x - x') + z$ lies in $\langle F \rangle + P$.

(iii) To prove (iii), we invoke (ii) to assert that $H(F) = \langle F \rangle \oplus F_+^\perp \cap -F_+^\perp$. Thus, we need to show $F_+^\perp \cap -F_+^\perp = \{0\}$. By hypothesis, x_{k+1}, \dots, x_N are precisely the points labeling the extreme rays of P which are not annihilated by $U(F^\perp)$. In particular, then, $U(F^\perp)x_{k+1}, \dots, U(F^\perp)x_N$ span F_+^\perp . So, if x lies in $F_+^\perp \cap -F_+^\perp$, then we may write $x = \sum_{i=k+1}^N \lambda_i U(F^\perp)x_i = \sum_{i=k+1}^N -\mu_i U(F^\perp)x_i$ where the λ_i and μ_i are nonnegative. Subtracting, we find that $U(F^\perp)(\sum_{i=k+1}^N (\lambda_i + \mu_i)x_i) = 0$. Thus $\sum_{i=k+1}^N (\lambda_i + \mu_i)x_i$ lies in $\langle F \rangle \cap P = F$. Since F is a face, and no x_i belongs to F for $i > k$, we conclude that $\lambda_i = \mu_i = 0$, $i = 1, 2, \dots, k$, as was to be shown.

4.2.1 REMARK. It follows from the lemma that F_+^\perp is closed. This, of course, also follows from Corollary 2.5 of [23]. In general, however, if P is a cone in \mathbf{R}^n and F is a face, F_+^\perp need not be closed. Indeed, if P is the forward light cone in \mathbf{R}^3 , discussed parenthetically in paragraph 3.12, and if F is some ray on the boundary of P , then F^\perp is a plane tangent to P along the ray on the boundary of P orthogonal to F . In this case, F_+^\perp consists of an open half-plane together with the origin. We note for later purposes that nevertheless, part (ii) of the lemma remains valid without assuming that P is polyhedral. Finally, it should be noted that Klee proves in [23] that a cone P in \mathbf{R}^n is polyhedral if and only if its projection onto each (affine) subspace of \mathbf{R}^n is closed. This simply reflects the fact that each polyhedral cone, in contradistinction with other types of cones, has a positive lower bound on the angles between its extreme rays.

4.3 Since $H(F) = \langle F \rangle$ and $\mathbf{R}^n/H(F)$ may be identified with F^\perp , $F \in \mathfrak{F}$, the model for the maximal ideal space of the C*-algebra A generated by $\{1_P * f \mid f \in L^1(\mathbf{R}^n)\}$ may now be presented as

$$Y = \{(F, y) \in \mathfrak{F} \times \mathbf{R}^n \mid y = U(F^\perp)y\}.$$

(We omit the tildas from here on.) The formula

$$(F, y) \oplus t = (F, y + U(F^\perp)t), \quad (F, y) \in Y, t \in \mathbf{R}^n,$$

gives the action of \mathbf{R}^n on Y , and the space X is simply $\{(F, x) \in Y \mid x \in F_+^\perp\}$. For $k = 0, 1, 2, \dots, n$, we write Y_k (resp. X_k) for $\{(F, y) \mid F \in \mathfrak{F}_k\}$ (resp. $\{(F, x) \mid F \in \mathfrak{F}_k\}$) and we identify \mathbf{R}^n (resp. P) with Y_0 (resp. X_0) in the obvious way. With these identifications, a function $\varphi = 1_P * f, f \in L^1(\mathbf{R}^n)$ may be extended to all of Y by the formula

$$(1) \quad \varphi(F, y) = \int_{y + \langle F \rangle - P} f(t) dt, \quad (F, y) \in Y,$$

(cf. 3.13). It should be noted that if we decompose \mathbf{R}^n into the direct sum $\langle F \rangle \oplus F^\perp$ and if we effect a corresponding decomposition of Lebesgue measure on \mathbf{R}^n as the product of Lebesgue measure on $\langle F \rangle$ with that on F^\perp , then we may use Lemma 4.2 and Fubini's theorem to express $\varphi(F, y)$ as

$$\int_{y - F_+^\perp} \int_{\langle F \rangle} f(s, t) ds dt.$$

This observation and Proposition 3.11 should help clarify further the remarks of 3.8. Nevertheless, from the computational point of view, it seems preferable to utilize the initial expression for $\varphi(F, y)$ rather than this alternative.

4.4 We are about to prove a lemma which extends Proposition 3.11 and which reveals immediately how to define the topology on Y . First observe that the vector $e = x_1 + x_2 + \dots + x_N$, where the x_i label the extreme rays of P , is an order unit for the order on \mathbf{R}^n induced by $\text{Int } P$; i.e., for each $y \in \mathbf{R}^n$, there is a $\lambda \in \mathbf{R}$ such that $-\lambda e <_P y <_P \lambda e$. On account of this, the subset $\{\lambda e \mid \lambda \in \mathbf{R}\}$ is cofinal in the directed set \mathbf{R}^n determined by $\text{Int } P$. Secondly, note that for any $y \in \mathbf{R}^n$ and any face F of P , we may write

$$\varphi(F, U(F^\perp)y) = \int_{y + \langle F \rangle - P} f(t) dt$$

where f and φ are related by formula (1) of the preceding paragraph. This is simply because $y + \langle F \rangle = U(F^\perp)y + \langle F \rangle$.

LEMMA. Let $f \in L^1(\mathbf{R}^n)$ and define φ on Y by equation (1) of paragraph 4.3. Let K be a compact subset of \mathbf{R}^n and let $\varepsilon > 0$ be given. Then there is a $\lambda > 0$ such that for every face F of P and every t in $\text{Int}(F^\perp)$ such that $\lambda U(F)e <_F t$ it follows that

$$|\varphi(G, U(G^\perp)(y + t)) - \varphi(F, U(F^\perp)y)| < \varepsilon$$

for all $y \in K$ and all faces G of P contained in F .

PROOF. Since the number of faces is finite, we may restrict attention to one face F . From the remarks preceding the lemma $\varphi(G, U(G^\perp)(y + t))$ is obtained by integrating f over $y + t + \langle G \rangle - P$ which yields the same result as integrating f_y over $t + \langle G \rangle - P$ where $f_y(x) = f(x - y)$. But for any t in $\langle F \rangle$, we have the inclusion

$$t - P \subseteq t + \langle G \rangle - P \subseteq \langle F \rangle - P.$$

Since the sets $t - P$ expand in a monotonic fashion to $\langle F \rangle - P$ as t increases along the directed set $\langle F \rangle$, the integrals of f_y over the sets $t + \langle G \rangle - P$ tend uniformly in G to the integral of f_y over $\langle F \rangle - P$ which is $\varphi(F, y)$. But the convergence is uniform in y , too, so long as y remains in K , by virtue of the continuity of translation on $L^1(\mathbf{R}^n)$. Finally, since $\{\lambda e \mid \lambda \in \mathbf{R}\}$ is cofinal in \mathbf{R}^n , $\{\lambda U(F)e \mid \lambda \in \mathbf{R}\}$ is cofinal in $\langle F \rangle$. The existence of λ therefore follows.

4.5 On the basis of Lemma 4.4, we are led to propose this candidate for the topology on Y : The topology is to have a basis consisting of the sets $\mathcal{V}_{F, \emptyset}$ where F runs through the faces of P , \emptyset runs through the open sets of \mathbf{R}^n , and where $\mathcal{V}_{F, \emptyset}$ is defined to be $\{(G, U(G^\perp)(y + U(F)t)) \mid G \subseteq F, y \in \emptyset, t \in \text{Int } P\}$. Lemma 4.4 shows that each φ is continuous on Y with respect to this topology and it is evident that translation on Y is continuous. Since each basic open set meets Y_0 in a set which is open when Y_0 is given the topology from \mathbf{R}^n , it follows that Y_0 is a dense open set in Y .

The topology on Y may seem a bit unwieldy at first sight. However, a moment's reflection will reveal that it does indeed generalize the usual topology on $(-\infty, \infty]$. In this case, $\{0\}$ and P are the only faces to the cone $P = [0, \infty)$ in \mathbf{R} , and so $Y = (\{0\}, y) \mid y \in \mathbf{R} \cup \{(P, 0)\}$ which may be identified with $(-\infty, \infty]$ in the obvious way. If \emptyset is the open interval (a, b) , say, then $\mathcal{V}_{\{0\}, \emptyset} = \{(\{0\}, y) \mid y \in (a, b)\}$ while $\mathcal{V}_{P, \emptyset} = \{(\{0\}, y) \mid y \in (a, \infty)\} \cup \{(P, 0)\}$. Thus, we do get the usual topology on $Y_0 = \{(\{0\}, y) \mid y \in \mathbf{R}\}$ and the neighborhoods of $(P, 0) (= \infty)$ are what they should be. (Further illustrative discussion will be found in 4.6.2.)

4.6 THEOREM. With respect to the topology just defined, Y is a locally compact Hausdorff space and the action of \mathbf{R}^n on Y is continuous. The map $1_P * f \rightarrow \varphi$, where $f \in L^1(\mathbf{R}^n)$ and φ is defined by equation (1) of paragraph 4.3, extends to an equivariant, *-isomorphism from A , the C^* -algebra generated by the $1_P * f$, onto $C_0(Y)$. If σ is the homeomorphism from Y to the maximal ideal space of A which is dual to this isomorphism, and if τ is the imbedding of \mathbf{R}^n into the maximal ideal space of A defined in paragraph 3.2, then $\sigma^{-1} \circ \tau$ is simply the identification of \mathbf{R}^n with Y_0 , $t \rightarrow (\{0\}, t)$. The image of P under $\sigma^{-1} \circ \tau$ is X_0 , while the closure of X_0 is X and is compact.

Thus Y is a model for the maximal ideal space of A and X is a model for the closure of $\tau(P)$.

4.6.1 PROOF. We first prove directly that X is compact. Since translates of the interior of X , which is $\{(F, x) \mid x \in \text{Int}(F^\perp)\}$, form a cover of Y , it will follow that Y is locally compact.

To this end, recall that $X = \bigcup_{k=0}^n X_k$ and that $X_k = \{(F, x) \mid F \in \mathfrak{F}_k, x \in F_+^\perp\}$. One should view X as stratified by the X_k . For any face F we shall write $\hat{F} = \{(F, x) \mid x \in F_+^\perp\}$ so that in particular, $\{0\}^\wedge$ is simply X_0 . By definition of the topology on Y , X_k in the *relative topology* is simply the disjoint union of the open sets \hat{F} , $F \in \mathfrak{F}_k$, and the topology on each of these is the Euclidean topology on F_+^\perp . The closure of each \hat{F} in X is simply the union $\bigcup_{F \subseteq G} \hat{G}$ as may be seen from the fact that if $\hat{F} \cap \mathcal{V}_{H,0}$ is nonempty, then $F \subseteq H$. Sets of the form $\mathcal{V}_{F, B(y, r)}$ where $B(y, r)$ is the ball of radius r centered at y also form a base for the topology on Y and to show that X is compact, it suffices to show that from any cover of X consisting of sets of this form, one may extract a finite subcover. So suppose \mathcal{U} is such a cover. Then \mathcal{U} contains a set of the form $\mathcal{V}_0 = \mathcal{V}_{P, B(y_0, r_0)}$ —a neighborhood of $(P, 0)$, the point at infinity. If $\hat{F} \subseteq X_{n-1}$, then

$$\hat{F} \cap \mathcal{V}_0 = \{(F, x) \mid x \in U(F^\perp)(B(y_0, r_0) + \text{Int } P)\}$$

which contains a set of the form $\{(F, x) \mid x \in U(F^\perp)(\lambda e + \text{Int } P)\}$ where λ is such that $x <_P \lambda e$ for all x in $B(y, r)$. Since F^\perp is one-dimensional, it follows that the complement of \mathcal{V}_0 in \hat{F} , $\hat{F} \setminus \mathcal{V}_0$, is contained in $\{(F, x) \mid x \in F_+^\perp, x \leq \lambda U(F^\perp)e\}$ ⁴ which is clearly a compact subset of X . Hence $X_{n-1} \setminus \mathcal{V}_0$ is also compact in X . If we cover $X_{n-1} \setminus \mathcal{V}_0$ by finitely many sets from \mathcal{U} , then it's not difficult to see that they cover all but a compact set in each \hat{F} , $F \in \mathfrak{F}_{n-2}$. Consequently we may find finitely many more sets in \mathcal{U} which cover $X_{n-2} \cup X_{n-1} \cup X_n$. Quite generally, then, what needs to be shown is that if $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_N$ covers $X_k \cup X_{k+1} \cup \dots \cup X_n$ where $\mathcal{V}_j = \mathcal{V}_{F_j, B(y_j, r_j)}$ (the F_j need not be distinct), then for F in \mathfrak{F}_{k-1} , $\hat{F} \setminus \mathcal{V}_0 \cup \dots \cup \mathcal{V}_N$ is compact in X . But if λ is such that $x <_P \lambda e$ for all x in $\bigcup_{j=0}^N B(y_j, r_j)$, then we'd like to assert that $\hat{F} \setminus (\mathcal{V}_0 \cup \mathcal{V}_1 \cup \dots \cup \mathcal{V}_N) \subseteq \{(F, x) \in \hat{F} \mid x \leq \lambda U(F^\perp)e\}$ which, manifestly, is compact in X . The details of the proof of this plausible assertion are rather cumbersome. So we content ourselves with a brief outline and an illustration.

We begin by reducing to the case when $F = \{0\}$ so that what we want to prove becomes the statement that if $X_1 \cup X_2 \cup \dots \cup X_n$ is covered by the \mathcal{V}_i , then the complement of their union in $\{0\}^\wedge = X_0$ is compact. Such a reduction is evident once it is recognized that the faces of F_+^\perp are precisely the sets $U(F^\perp)H$ where H runs through the faces of P containing F . (This may be proved along the lines of Proposition 2.6 of [23].) So, we assume that $F = \{0\}$ and we let G_k be the face generated by x_k . Since each \hat{G}_k is contained in $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \dots \cup \mathcal{V}_N$, it follows that $U(G^\perp)P$ is contained in $U(G^\perp)(\bigcup_{i=1}^N B(y_i, r_i) + U(F_j)P)$. But then it follows that

⁴The order here is that on F^\perp determined by F_+^\perp . We omit the subscript in order to streamline the notation.

$X_0 \cap (\mathcal{V}_0 \cup \mathcal{V}_1 \cup \dots \cup \mathcal{V}_N)$ contains $(\lambda U(G_k)e) + \text{Int } P$ for each k , and this means that $X_0 \setminus \mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots \cup \mathcal{V}_N \subseteq \{(\{0\}, x) \in X_0 \mid x \leq_P \lambda e\}$. This substantially completes the proof that X is compact.

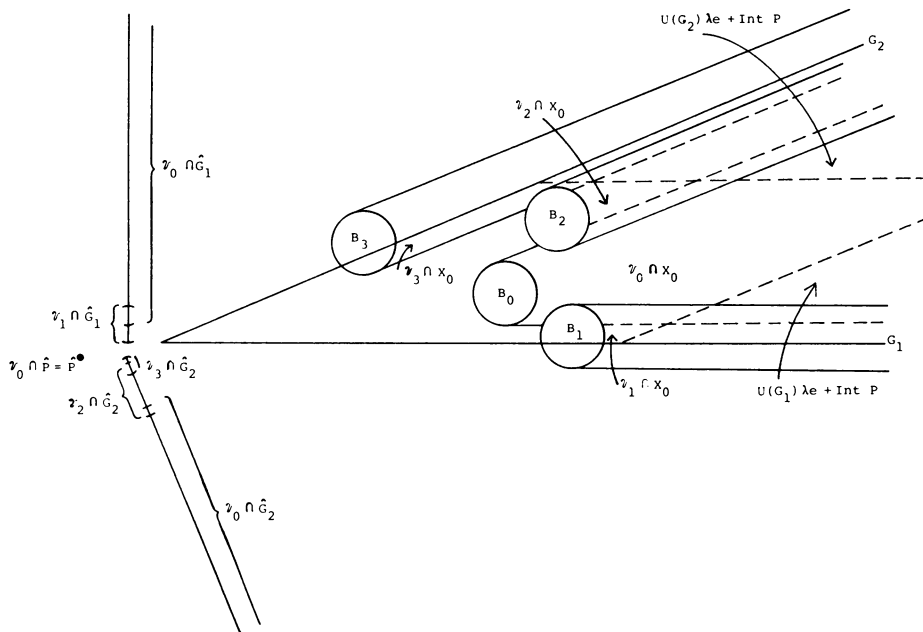


FIGURE 4.1

4.6.2 We interrupt the proof momentarily to discuss Figure 4.1 which illustrates what happens for a cone P in \mathbb{R}^2 . We have purposely chosen a cone which is not self-dual so that the reader might avoid certain pitfalls associated with thinking of all cones as having right angles between faces. We have identified $X_0 = \{0\}^\wedge$ with P . Similarly \hat{G}_1 and \hat{G}_2 are identified with G_1^\perp and G_2^\perp respectively and $X_1 = \hat{G}_1 \cup \hat{G}_2$. Finally, we identify $\hat{P} = \{(P, 0)\} = X_2$ with the point at infinity and place it near the apex of P . We have drawn four basic open sets, $\mathcal{V}_0, \dots, \mathcal{V}_3$, which cover $X_1 \cup X_2$; \mathcal{V}_0 is a neighborhood of $(P, 0)$, $\mathcal{V}_1 \cap \hat{G}_2$ covers what $\mathcal{V}_0 \cap \hat{G}_2$ misses, and $(\mathcal{V}_2 \cup \mathcal{V}_3) \cap \hat{G}_1$ covers what $\mathcal{V}_0 \cap \hat{G}_1$ misses. Each \mathcal{V}_i is determined by a certain ball B_i ; $\mathcal{V}_0 \cap X_0 = B_0 + \text{Int } P$, $\mathcal{V}_1 \cap X_0 = B_1 + U(G_1)\text{Int } P$, $\mathcal{V}_2 \cap X_0 = B_2 + U(G_2)\text{Int } P$, and $\mathcal{V}_3 \cap X_0 = B_3 + U(G_2)\text{Int } P$. Observe that $X_0 \cap (\mathcal{V}_0 \cup \dots \cup \mathcal{V}_3)$ contains $(U(G_1)\lambda e + \text{Int } P) \cup (U(G_2)\lambda e + \text{Int } P)$ so that what is in X_0 but not in $\mathcal{V}_0 \cup \dots \cup \mathcal{V}_3$ is all dominated by λe ; i.e., what's left over is compact.

4.6.3 Continuing with the proof, recall that we have already noted that each φ is continuous on Y and of course φ is bounded by the L^1 -norm of the function f which determines it. Moreover, given $\varepsilon > 0$, if we choose $t \in \mathbb{R}^n$ such that $\int_{\mathbb{R}^n \setminus t+P} |f| dt < \varepsilon$ (and such a choice certainly is possible), then by definition of φ , it follows that $|\varphi(F, y)| < \varepsilon$ for all $(F, y) \in Y \setminus X \oplus t$. Since $X \oplus t$ is compact, this shows that $\varphi \in C_0(Y)$.

Next we verify that the functions φ separate the points of Y . If (F, y) and (G, x) are identified by every φ , then $y + \langle F \rangle - P = x + \langle G \rangle - P$ a.e. By regularity, $y + \langle F \rangle - P = x + \langle G \rangle - P$. By Lemma 4.2, then, $(-y) + F_+^\perp \oplus \langle F \rangle = (-x) + G_+^\perp \oplus \langle G \rangle$. Hence $F_+^\perp = (y - U(F^\perp)x) + U(F^\perp)G_+^\perp + U(F^\perp)\langle G \rangle$. By Lemma 4.2 again, $F_+^\perp \cap -F_+^\perp = \{0\}$, and this, in turn, implies that F_+^\perp contains no affine subspace (of positive dimension). (Perhaps the easiest way to prove this is to appeal to Theorem 2.3 of [5] which asserts that a closed cone C in \mathbf{R}^n satisfies $C \cap -C = \{0\}$ if and only if its dual cone, $C^\dagger = \{x \mid x \cdot y \geq 0, y \in C\}$, has nonempty interior. Since the dual of any cone containing a nontrivial affine subspace obviously can't have a nonempty interior, our assertion follows.) Hence $U(F^\perp)\langle G \rangle = 0$ and, in a like manner, $U(G^\perp)\langle F \rangle = 0$. Thus $\langle F \rangle = \langle G \rangle$. But then, $F = \langle F \rangle \cap P = \langle G \rangle \cap P = G$; $x, y \in F^\perp$; and $-x + F_+^\perp = -y + F_+^\perp$. Since $F_+^\perp \cap -F_+^\perp = \{0\}$, once more, we deduce that $x = y$. Finally, we remark that it is clear that given $(F, y) \in Y$, one can find an f in L^1 such that the associated φ doesn't vanish at (F, y) . Thus by the Stone-Weierstrass theorem, the C^* -algebra generated by the functions φ is all of $C_0(Y)$.

Since Y_0 is dense in Y and since the restriction of φ to Y_0 is essentially $1_P * f$ on \mathbf{R}^n , we conclude that the map $1_P * f \rightarrow \varphi$ does indeed extend to a $*$ -isomorphism from A onto $C_0(Y)$ which is obviously equivariant. Since the other assertions of the theorem are now obvious and require no further comment, we conclude that the proof is complete.

4.7 The groupoid \mathcal{G} whose representation theory yields that of $\mathfrak{B}(P)$ is, by virtue of Theorem 4.6, simply $Y \times \mathbf{R}^n \mid X$. The Haar system which we use is $\{\lambda^{(F,x)}\}_{(F,x) \in X}$ where $\lambda^{(F,x)}$ is the restriction to \mathcal{G} of the measure $\delta_{(F,x)} \times \lambda$, with λ denoting Lebesgue measure on \mathbf{R}^n . Since \mathbf{R}^n is commutative, and therefore amenable, we know by Proposition 2.15 and Theorem 3.7 that W , the Wiener-Hopf representation of $C^*(\mathcal{G})$, is faithful and effects an isomorphism between $C^*(\mathcal{G})$ and $\mathfrak{B}(P)$. We know, too, from the definition of the topology on X that X_0 , which is essentially P , is open so that X is a regular compactification of P . Hence, by Corollary 3.7.2, $\mathfrak{B}(P)$ contains $\mathcal{K} = \mathcal{K}(L^2(P))$. More generally, let $\tilde{X}_k = \bigcup_{j=0}^k X_j$, $k = 0, 1, \dots, n$. Then each \tilde{X}_k is an open invariant subset of the unit space of \mathcal{G} and by Proposition 2.16 (together with W) we may identify $C^*(\mathcal{G} \mid \tilde{X}_k)$ with a closed, two-sided ideal I_k in $\mathfrak{B}(P)$. By Corollary 3.7.2 and its proof, I_0 is \mathcal{K} .

We summarize these facts and more in the following theorem which is the principal objective of this section.

THEOREM. (i) *The C^* -algebra $\mathfrak{B}(P)$ is postliminal and contains $\mathcal{K}(L^2(P)) = I_0$.*

(ii) *The ideals I_k , $k = 0, 1, \dots, n$, form a composition series for $\mathfrak{B}(P)$. The quotient I_k/I_{k-1} , $k = 0, 1, \dots, n-1$, is isomorphic to $C_0(\mathfrak{F}_k \times \mathbf{R}^k) \otimes \mathcal{K}$, where \mathfrak{F}_k is given the discrete topology, $\mathbf{R}^0 = \{0\}$, and where $I_{-1} = \{0\}$. The quotient I_n/I_{n-1} is isomorphic to $C_0(\mathbf{R}^n)$.*

(iii) *As a set, the spectrum of $\mathfrak{B}(P)$, $\text{spec}(\mathfrak{B}(P))$, may be identified with $\bigcup_{k=0}^n \mathfrak{F}_k \times \mathbf{R}^k$, and the sets $(\{F\} \times \emptyset) \cup (\bigcup_{j=0}^{k-1} \mathfrak{F}_j \times \mathbf{R}^j)$ where $F \in \mathfrak{F}_k$ and \emptyset is open in \mathbf{R}^k form a basis for the topology on $\text{spec}(\mathfrak{B}(P))$.*

PROOF. Assertion (i) follows from (ii) as does the identification $\text{spec}(\mathfrak{B}(P)) = \bigcup_{k=0}^n \mathfrak{F}_k \times \mathbb{R}^k$. For the assertion regarding the topology, recall that we may identify the spectrum of I_k with an open subset of $\text{spec}(\mathfrak{B}(P))$ by [11, 3.2.2]. So the topology on $\text{spec}(\mathfrak{B}(P))$ has a base of the indicated form once it is noted that $\text{spec}(I_0) = \mathfrak{F}_0 \times \mathbb{R}^0$, a single point, is dense $\text{spec}(\mathfrak{B}(P))$. But this follows from the fact that $\mathfrak{B}(P)$ has a faithful, irreducible representation—the identity representation (cf. [11, 3.9.1]). Thus all that remains is to show that I_k/I_{k-1} is isomorphic to $C_0(\mathfrak{F}_k \times \mathbb{R}^k) \otimes \mathcal{K}$.

By Proposition 2.16, I_k/I_{k-1} is isomorphic to $C^*(\mathfrak{G}|X_k)$. Since the space X_k , in the relative topology for X , is the disjoint union of the open sets \hat{F} , $F \in \mathfrak{F}_k$, and since these are invariant, I_k/I_{k-1} is isomorphic to the direct sum, taken over all faces F in \mathfrak{F}_k , of the algebras $C^*(\mathfrak{G}|\hat{F})$. Hence, it suffices to show that $C^*(\mathfrak{G}|\hat{F})$ is isomorphic to $C_0(\mathbb{R}^k) \otimes \mathcal{K}$. Now the groupoid $\mathfrak{G}|\hat{F}$ is simply $\{(F, x, t) \mid x \in F_+^\perp, t \in \mathbb{R}^n, x + U(F^\perp)t \in F_+^\perp\}$ and it is evident from this that the isotropy group of each point in \hat{F} is $\langle F \rangle$ which, in turn, is isomorphic to \mathbb{R}^k . If we decompose Lebesgue measure on \mathbb{R}^n into the product of Lebesgue measure on $\langle F \rangle$ with that on F^\perp then with the obvious choices of Haar systems corresponding to this decomposition, it is clear that the map

$$(F, x, t) \rightarrow ((x, x + U(F^\perp)t), U(F)t)$$

sets up a topological isomorphism between $\mathfrak{G}|\hat{F}$ and the product of the trivial groupoid on F_+^\perp with the group $\langle F \rangle$ (cf. 2.3 and 3.7.1). Since the C^* -algebra of the latter groupoid is clearly $C^*(\langle F \rangle) \otimes \mathcal{K} \cong C_0(\langle F \rangle) \otimes \mathcal{K} \cong C_0(\mathbb{R}^k) \otimes \mathcal{K}$, the proof is complete.

4.7.1 The following corollary generalizes a well-known result for Wiener-Hopf operators in one dimension and is related to a theorem of Coburn and Douglas [8, Theorem 2] about Wiener-Hopf operators with almost periodic symbol on a locally compact abelian group determined by a subsemigroup. The proof is immediate.

COROLLARY. *The commutator ideal of $\mathfrak{B}(P)$ is I_{n-1} and the quotient $\mathfrak{B}(P)/I_{n-1}$ is isomorphic to $C_0(\mathbb{R}^n)$.*

4.7.2 COROLLARY. *The C^* -algebra $\mathfrak{B}(P)$ is solvable of length n .*

PROOF. It follows from the theorem that $\mathfrak{B}(P)$ is solvable and that its length is at most n . But from the description of the topology on $\text{spec}(\mathfrak{B}(P))$ and the one-to-one, order preserving correspondence between ideals and open subsets of $\text{spec}(\mathfrak{B}(P))$ [11, 3.2.2], it is clear that the length is n . In fact it is clear that the I_k 's furnish the only composition series of length n .

4.7.3 In one dimension, the theorem recaptures the discussion in §V of [7].

4.8 We conclude this section with two remarks. First of all, from Theorem 4.6 and its proof one may deduce a very explicit expression for the irreducible representations of $\mathfrak{B}(P)$ as follows. Let L be such a representation and let k be the first integer such that the restriction of L to I_k is nonzero. Then by [11, 2.10.4], L is uniquely determined by this restriction. Since L annihilates I_{k-1} , we may view L as an irreducible representation of $I_k/I_{k-1} \cong C^*(\mathfrak{G}|X_k)$. Since, as we have seen in the

proof of Theorem 4.6, X_k is the disjoint union of the open invariant sets \hat{F} , $F \in \mathfrak{F}_k$, we may appeal to Proposition 2.16 to write $C^*(\mathfrak{G} | X_k) = \sum_{F \in \mathfrak{F}_k} \oplus C^*(\mathfrak{G} | \hat{F})$, and so L is uniquely determined by its restriction to some $C^*(\mathfrak{G} | \hat{F})$, $F \in \mathfrak{F}_k$. As we have also seen, $\mathfrak{G} | \hat{F}$ is isomorphic to the product of the group $U(F)\mathbf{R}^n$ with the reduction to F_+^\perp of the transformation group obtained by letting $U(F^\perp)\mathbf{R}^n$ act on $U(F^\perp)\mathbf{R}^n$. This enabled us to conclude that $C^*(\mathfrak{G} | \hat{F})$ is isomorphic to $C_0(U(F)\mathbf{R}^n) \otimes \mathcal{K}$. Thus L is completely determined by F and an element c of $U(F)\mathbf{R}^n$. That is, L is equivalent to the tensor product of $\text{Ind } \delta_0$ on $C^*(U(F^\perp)\mathbf{R}^n \times U(F^\perp)\mathbf{R}^n | F_+^\perp)$ and the character of $C_0(U(F)\mathbf{R}^n)$ determined by c . From the form of the isomorphism, we can conclude upon retracing our steps that L on $C^*(\mathfrak{G})$ is equivalent to $L^{(F,c)}$ defined on $L^2(F_+^\perp)$ by the formula

$$\begin{aligned} L^{(F,c)}(f)\xi(x) &= \int_{\mathbf{R}^n} f(F, x, t) 1_{F_+^\perp}(x + U(F^\perp)t) \\ &\quad \times \exp(-i\langle U(F)t, c \rangle) \xi(x + U(F^\perp)t) d\lambda(t) \end{aligned}$$

where $f \in C_c(\mathfrak{G})$, $\xi \in L^2(F_+^\perp)$, and where λ denotes Lebesgue measure on \mathbf{R}^n . It is evident that two different pairs (F_1, c_1) and (F_2, c_2) give inequivalent representations since the representations have different kernels, and so we have parameterized explicitly the spectrum of $\mathfrak{B}(P)$.

For our second remark, we note that in the course of this investigation we have been able to invoke 3.7.1 in order to avoid explicit discussion of Wiener-Hopf operators on cones P such that $P \cap -P$ contains more than just 0. However, such operators arise quite naturally in the study of Wiener-Hopf operators on a cone containing no lines, and we would like to indicate briefly how. Suppose then that P is such a cone and keep the notation of the preceding paragraphs. We have considered the consecutive quotients I_k/I_{k-1} coming from the composition series for $\mathfrak{B}(P)$, but now we want to consider the quotients $\mathfrak{B}(P)/I_{k-1}$, $k = 1, 2, \dots, n$. By Proposition 2.16, each such quotient may be identified with

$$C^*(\mathfrak{G} | (X_k \cup X_{k+1} \cup \dots \cup X_n)).$$

As we observed in 4.6.1, for each $F \in \mathfrak{F}_k$, the closure of \hat{F} is $\bigcup_{F \subseteq G} \hat{G}$ —a closed invariant set contained in $X_k \cup X_{k+1} \cup \dots \cup X_n$. By Proposition 2.16, again, we may think of $C^*(\mathfrak{G} | \hat{F}^{\text{cl}})$ as a quotient of $\mathfrak{B}(P)/I_{k-1}$. But it's quite straightforward to see that $\mathfrak{G} | \hat{F}^{\text{cl}}$ is topologically isomorphic to the groupoid constructed *ab initio* from the polyhedral cone $\langle F \rangle + P$ —a cone which contains the subspace $\langle F \rangle$. Thus we find that for each face F of P , $\mathfrak{B}(P)$ has a quotient isomorphic to $\mathfrak{B}(\langle F \rangle + P)$. On the other hand, since the sets \hat{F}^{cl} cover $X_k \cup X_{k+1} \cup \dots \cup X_n$ as F runs over \mathfrak{F}_k , we see that the map $f \rightarrow \sum_{F \in \mathfrak{F}_k} f_F$ from $C_c(\mathfrak{G} | (X_k \cup X_{k+1} \cup \dots \cup X_n))$ to $\sum_{F \in \mathfrak{F}_k} \oplus C_c(\mathfrak{G} | \hat{F}^{\text{cl}})$, where f_F denotes the restriction of f to $\mathfrak{G} | \hat{F}^{\text{cl}}$, extends to a C^* -injection of $C^*(\mathfrak{G} | (X_k \cup X_{k+1} \cup \dots \cup X_n))$ into $\sum_{F \in \mathfrak{F}_k} \oplus C^*(\mathfrak{G} | \hat{F}^{\text{cl}})$. Thus $\mathfrak{B}(P)/I_{k-1}$ may be imbedded canonically in $\sum_{F \in \mathfrak{F}_k} \mathfrak{B}(\langle F \rangle + P)$. In particular, if one wants to study the Fredholm properties of operators in $\mathfrak{B}(P)$ (or more appropriately, in $\mathfrak{B}(P)$ with the identity adjoined), one wants to understand $\mathfrak{B}(P)/I_0$, since $I_0 = \mathcal{K}$. Thus one is led to study the spectral theory of operators in

$\mathfrak{B}(\langle F \rangle + P)$ as F ranges over the one-dimensional faces of P . We see, then, that a natural question about Wiener-Hopf operators on a cone containing no lines leads ineluctably to cones of the opposite type. Of course at this point one probably would want to invoke 3.7.1 and 4.2 to identify $\mathfrak{B}(\langle F \rangle + P)$ with $\mathfrak{B}(F_+^\perp) \otimes C_0(\langle F \rangle)$. When P is the standard orthant, these observations were first made by Douglas and Howe in [13].

5. Cones and Jordan algebras. Let P be a closed cone in \mathbf{R}^n and let P^\dagger denote its dual, $\{x \in \mathbf{R}^n \mid x \cdot y \geq 0, y \in P\}$. Evidently, P^\dagger is also a closed cone, and one says that P is *self-dual* if $P = P^\dagger$. If P is self-dual, then P is the closure of its interior $\text{Int}(P)$, and $\text{Int}(P) = \{x \in \mathbf{R}^n \mid x \cdot y > 0, y \in P \setminus \{0\}\}$ (cf. [5, pp. 8, 9]). One says that a cone P , which is the closure of its interior, is *homogeneous* if the collection of invertible linear transformations on \mathbf{R}^n which map P into P acts transitively on $\text{Int}(P)$. Homogeneous, self-dual cones, otherwise known in the literature as *domains of positivity*, play a central role in the theory of functions of several complex variables. This theory, in turn, is related to the present work by the fact that the Fourier transform converts a Wiener-Hopf operator defined over a cone P into a Toeplitz operator defined over the tube domain based on P . Indeed many people prefer to study Wiener-Hopf operators from this perspective (cf. [3, 4, 8, 9, 13]). Fortunately for everyone concerned, the homogeneous, self-dual cones have all been identified. It turns out that each such cone is uniquely associated with a finite dimensional, formally real, Jordan algebra and the entire structure of the cone, including its facial structure, can easily be read out of the algebraic structure of the associated Jordan algebra. Therefore, in order to carry out our analysis of the C^* -algebra of Wiener-Hopf operators defined over a homogeneous, self-dual cone, we need to digress momentarily in order to collect the basic facts from the theory of Jordan algebras which we will use. For the structure of Jordan algebras we refer primarily to the book of Braun and Koecher [6] and to the very readable recent article by Alfsen, Shultz, and Störmer [1]. The connection between Jordan algebras and homogeneous self-dual cones is discussed at great length in Koecher's notes [24] and in Kapitel XI of [6]. Finally, for an introductory survey of the role of these cones in complex analysis, the reader should consult Vagi's article in [2].

5.1 By definition a *formally real* (finite dimensional) *Jordan algebra* is a finite dimensional commutative, but nonassociative, algebra $(\mathfrak{A}, +, \circ)$ over \mathbf{R} such that (a) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathfrak{A}$, where $x^2 = x \circ x$, and (b) if $\sum_{i=1}^n x_i^2 = 0$, then $x_i = 0$ for all i . Since these are the only type of Jordan algebra we consider, we shall drop the adjectives “formally real” and “finite dimensional” and simply use the term “Jordan algebra.”

It turns out that such an algebra has an identity which we denote by $\mathbf{1}$ and that it is semisimple; i.e., it is the direct sum of a finite number of Jordan algebras each having no nontrivial ideals. A Jordan algebra with only trivial ideals is, of course, called *simple*. The simple Jordan algebras were catalogued by Jordan, von Neumann, and Wigner in the famous paper [22]. Here is the list; the notation will be explained subsequently.

- (i) \mathbf{R} ,
- (ii) \mathfrak{S}_n , $n \geq 3$,
- (iii) $\mathfrak{H}_n(\mathbf{R})$, $n \geq 3$,
- (iv) $\mathfrak{H}_n(\mathbf{C})$, $n \geq 3$,
- (v) $\mathfrak{H}_n(\mathbf{Q})$, $n \geq 3$,
- (vi) $\mathfrak{H}_3(\mathbf{O})$.

Of course \mathbf{R} has its usual operations and is the only simple, *associative* Jordan algebra. The algebras \mathfrak{S}_n are called *spin factors*; the terminology is due to Topping [29]. The underlying vector space structure of \mathfrak{S}_n is \mathbf{R}^n and its identity $\mathbf{1}$ is the vector $(0, 0, \dots, 1)$. If we write the elements of \mathfrak{S}_n in the form $\alpha\mathbf{1} + u$ where u is orthogonal to $\mathbf{1}$, then the multiplication on \mathfrak{S}_n may be written in this way: $(\alpha\mathbf{1} + u) \circ (\beta\mathbf{1} + v) = (\alpha\beta + u \cdot v)\mathbf{1} + (\beta u + \alpha v)$ where $u \cdot v$ denotes the usual inner product of u and v . In (iii)–(vi), \mathbf{R} and \mathbf{C} have their usual meanings; \mathbf{Q} denotes the division algebra of quaternions; and \mathbf{O} denotes the Cayley numbers, an eight dimensional, nonassociative division algebra over \mathbf{R} . There is a common perspective from which one can view all of these division algebras, starting with \mathbf{R} . We let \mathbf{R} have the identity involution which we denote by $*$. If we know how to construct one of the division algebras \mathbf{K} in the list, together with an involution also denoted by $*$, then we construct the next one as the set of ordered pairs (a, b) , $a, b \in \mathbf{K}$. The product of two such pairs is given by the formula $(a, b)(c, d) = (ac - d*b, da + bc*)$ (note the order in the products occurring in the right-hand side), and $(a, b)*$ is defined to be $(a*, -b)$. Beginning with \mathbf{R} and the identity involution, this construction is familiar and yields \mathbf{C} . The construction applied to \mathbf{C} yields \mathbf{Q} and when applied to \mathbf{Q} , it yields \mathbf{O} . It should be noted that \mathbf{C} and \mathbf{Q} are associative precisely because \mathbf{R} and \mathbf{C} are commutative; \mathbf{O} is not associative because \mathbf{Q} is not commutative. If \mathbf{K} denotes any of these division algebras over \mathbf{R} , then $\mathfrak{H}_n(\mathbf{K})$ denotes the collection of all $n \times n$ matrices (a_{ij}) over \mathbf{K} such that $(a_{ij}) = (a_{ji}^*)$; i.e., $\mathfrak{H}_n(\mathbf{K})$ is the collection of Hermitian $n \times n$ matrices over \mathbf{K} . Addition in $\mathfrak{H}_n(\mathbf{K})$ is the usual componentwise addition and the product is given by the formula $x \circ y = \frac{1}{2}(xy + yx)$ where xy denotes the usual matrix product of x and y . In each $\mathfrak{H}_n(\mathbf{K})$, the identity is the usual matrix identity.

It should be noted that for each of the division algebras \mathbf{K} , $\mathfrak{H}_1(\mathbf{K})$ is isomorphic to \mathbf{R} and that $\mathfrak{H}_2(\mathbf{K})$ can be realized as an \mathfrak{S}_n for a suitable n . Note, too, that \mathfrak{S}_1 is \mathbf{R} while \mathfrak{S}_2 is isomorphic to $\mathbf{R} \oplus \mathbf{R}$ and so is not simple. Finally we note that if $n > 3$, $\mathfrak{H}_n(\mathbf{O})$ is not a Jordan algebra. These observations explain the limits on the parameter n in each item of the list.

Each of the algebras in categories (i)–(v) can be faithfully represented as a Jordan algebra of Hermitian matrices on a finite dimensional complex Hilbert space; $\mathfrak{H}_3(\mathbf{O})$ can't. Accordingly, Jordan algebras which can be so represented are called *special*; $\mathfrak{H}_3(\mathbf{O})$ is referred to as *the exceptional Jordan algebra*. One of the main difficulties with formulating the theory of Jordan algebras is to find a way of doing things in a unified fashion, covering the special and the exceptional algebras simultaneously.

5.2 Let \mathfrak{A} be a Jordan algebra and for $a \in \mathfrak{A}$, let $L(a)$ denote the linear transformation on \mathfrak{A} defined by the formula $L(a)x = a \circ x$, $x \in \mathfrak{A}$. Although it is something of a misnomer, L is called the regular representation of \mathfrak{A} on \mathfrak{A} . It is not

actually a homomorphism; $L(u \circ v)$ need not equal $L(u)L(v)$. The *Killing form* on \mathfrak{A} is, by definition, the bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{A} given by the formula $\langle x, y \rangle = \text{tr}(L(x \circ y))$. (Here, tr denotes the usual trace defined on the linear transformations over \mathfrak{A} .) This form is clearly symmetric. It is also positive definite and so, with respect to it, \mathfrak{A} becomes an Euclidean space. There is another representation of \mathfrak{A} on \mathfrak{A} , the so-called *quadratic representation*, which is important in the theory and plays a central role here. It is denoted by U and is defined by the formula $U(a)x = 2a \circ (a \circ x) - a^2 \circ x$, $a, x \in \mathfrak{A}$; equivalently, $U(a) = 2(L(a))^2 - L(a^2)$. If \mathfrak{A} is special and realized as Hermitian matrices on a finite dimensional Hilbert space, then $U(a)x = axa$ where the product on the right is the usual matrix product.

5.3 LEMMA. (i) *With respect to the Killing form on \mathfrak{A} , $L(a)$ and $U(a)$ are selfadjoint for every $a \in \mathfrak{A}$.*

(ii) *If e is an idempotent in \mathfrak{A} , i.e., if $e^2 = e$, then $U(e)$ is an orthogonal projection.*

PROOF. (i) The fact that $L(a)$ is selfadjoint is verified by somewhat tedious computation. The details are in Chapter III, Lemma 4, of [24]. The fact that $U(a)$ is selfadjoint follows from the selfadjointness of $L(a)$.

(ii) We need only note that $U(e)$ is idempotent; this is proved on p. 21 of [1] on the basis of Macdonald's theorem and directly in Satz 4, Kapitel III, of [6].

(If \mathfrak{A} is special, then it is evident that $U(e)$ is idempotent, for in this case $U(e)$ is the familiar "cornering operator." Macdonald's theorem asserts that if certain types of identities are valid in all special Jordan algebras, then they are valid in the exceptional one as well. Since the property of being idempotent can be expressed in terms of such identities, the fact that $U(e)$ is idempotent in the exceptional case is a consequence of the result for special Jordan algebras.)

5.4 Jordan algebras are power associative. This means that the subalgebra generated by a single element and the identity is associative. The *spectrum* of an element in a Jordan algebra is the familiar spectrum of the element in the algebra that it and the identity generate; i.e., the spectrum $\Lambda(a)$ of $a \in \mathfrak{A}$ is the set of roots of the minimal polynomial of a . It is not difficult to show that the spectrum of $L(a)$ is $\{\frac{1}{2}(\lambda + \mu) \mid \lambda, \mu \in \Lambda(a)\}$ and the spectrum of $U(a)$ is $\{\lambda\mu \mid \lambda, \mu \in \Lambda(a)\}$ [6, Satz 1.3, Kapitel VIII]. Since our algebras are formally real, we have a spectral theorem at our disposal. We may write each a in the form $a = \sum_{i=1}^n \lambda_i e_i$ where $\{\lambda_1, \dots, \lambda_n\}$ is the spectrum of a and the e_i are orthogonal idempotents, the *spectral projections* of a . As one might suspect, to say that two idempotents e and f are orthogonal is to say that $e \circ f = 0$. As it turns out, this is equivalent to the equation $U(e)U(f) = 0$ by Lemma 4.8 of [1]. The spectral projections of an element all lie in the subalgebra it generates with the identity (cf. [6, Hilfssatz 3.2, Kapitel XI] or [1, Proposition 4.3]).

5.5 LEMMA. *The following assertions about an element a in a Jordan algebra are equivalent.*

- (i) $\Lambda(a)$ contains nonnegative numbers only;
- (ii) a is a square;
- (iii) $L(a)$ is positive semidefinite with respect to the Killing form; and
- (iv) $U(a)$ is positive semidefinite with respect to the Killing form.

The proof is evident from what we've presented so far. For further elaboration the reader should consult §§3.4 and 3.5 in Kapitel XI of [6].

5.6 An element in a Jordan algebra is called *positive* if it satisfies any and hence all of the conditions of Lemma 5.5. From (iii) of the lemma, it is clear that the collection P of all positive elements is a closed cone and from (i) it is clear that $P \cap -P = \{0\}$.

LEMMA. (i) P is a self-dual cone.

(ii) $\text{Int } P = \{a \in P \mid 0 \notin \Lambda(a)\} = \{a \in P \mid a \text{ is invertible}\}$.

(iii) P is the closure of $\text{Int } P$.

(iv) P is homogeneous; in fact, $U(a)P \subseteq P$ for all $a \in \mathfrak{A}$, $U(a)$ is invertible for all invertible $a \in \mathfrak{A}$, and the group generated by the $U(a)$, with a invertible, acts transitively on $\text{Int } P$.

PROOF. Assertion (i) is proved in [6, Satz 3.8, Kapitel XI] while (ii) is proved in [6, Satz 3.6, Kapitel XI]. Assertion (iii) is obvious and (iv) is [6, Satz 4.5, Kapitel XI]. See Propositions 2.5 and 2.7 and Theorem 6.10 of [1] also.

5.7 For the record we record the fact that the map $\mathfrak{A} \rightarrow P$ sets up a bijection between formally real, finite dimensional Jordan algebras and homogeneous, self-dual cones. A proof of this may be found in Theorem 15, Chapter VI, of [24].

5.8 We now want to analyze the facial structure of P . If e is an idempotent in \mathfrak{A} , then we write e^\perp for $1 - e$. We want to emphasize that, as we shall see, $U(e^\perp)$ generally does not coincide with $I - U(e)$ which we shall denote by $U(e)^\perp$. (Here, “ I ” denotes the identity matrix on \mathfrak{A} .) Recall from 3.12 and 4.2.1 that if F is a face of an arbitrary closed cone P then $\langle F \rangle + P$ and the projection of P on F^\perp need not be closed. However, if we now let $\langle F \rangle + P$ stand for its *closure* and let F_+^\perp denote the *closure* of the projection of P on F^\perp , then we still have the equation $\langle F \rangle + P = \langle F \rangle \oplus F_+^\perp$ proved in Lemma 4.2 for polyhedral cones. *This we shall do for the remainder of the paper.* Recall, finally that $H(F) = \langle F \rangle + P \cap (\langle F \rangle - P)$.

5.9 LEMMA. (i) The map $e \rightarrow U(e)(P)$ from idempotents in \mathfrak{A} to subsets of P is a bijection between the idempotents and the faces of P .

(ii) If $F = U(e)P$ is a face of P , then the orthogonal complement of $H(F)$ in F_+^\perp is $U(e^\perp)P$; i.e., $F_+^\perp \ominus H(F) = U(e^\perp)P$.

PROOF. (i) Suppose first that e is an idempotent, that $x \in U(e)P$ and that $y \in P$ satisfies $x \geq y$. Then by Corollary 2.10 of [1], $0 = U(e^\perp)x \geq U(e^\perp)y \geq 0$. Corollary 2.10 of [1], again, shows that $U(e)y = y$, so that $y \in U(e)P$. Thus $U(e)P$ is a face. Suppose, conversely, that F is a face and, without loss of generality, suppose too that $F \neq \{0\}$. If a is a nonzero element of F , then it is easy to see that the spectral projections of a corresponding to nonzero points in the spectrum of a all belong to F . Consequently, F is spanned by the idempotents it contains. According to Proposition 4.9 of [1], the set of idempotents in a Jordan algebra is a complete orthomodular lattice in the ordering determined by P where the orthocomplement of an idempotent e is e^\perp . Since F is closed, it follows that the set of idempotents in F is

a complete sublattice and the identity for this sublattice is the supremum e of all idempotents in F . That is, $F = U(e)P$. Since $e = U(e)(\mathbf{1})$, the map $e \rightarrow U(e)P$ is bijective, and the proof of (i) is complete.

(ii) Since $\langle F \rangle + P = \langle F \rangle \oplus F_+^\perp$ by 5.8, $F_+^\perp \ominus H(F) = F_+^\perp \ominus L(F)$ where $L(F) = F_+^\perp \cap (-F_+^\perp)$. On the other hand, as a consequence of Corollary 2.2 of [5], we know that if C is an arbitrary closed cone in \mathbf{R}^n such that $C^\dagger \subseteq C$, then $C \ominus L(C) = C^\dagger$. Thus we need to show that $(F_+^\perp)^\dagger \cap F^\perp \subseteq F_+^\perp$ and that $(F_+^\perp)^\dagger \cap F^\perp = U(e^\perp)P$. By definition $F^\perp = U(e)^\perp \mathfrak{A}$ and F_+^\perp is the closure of $U(e)^\perp P$. So

$$\begin{aligned} (F_+^\perp)^\dagger \cap F^\perp &= \{x \in F^\perp \mid \langle x, y \rangle \geq 0 \text{ for all } y \in F_+^\perp\} \\ &= \{x \in F^\perp \mid \langle x, U(e)^\perp y \rangle \geq 0 \text{ for all } y \in P\} \\ &= \{x \in F^\perp \mid \langle U(e)^\perp x, y \rangle \geq 0 \text{ for all } y \in P\} \\ &= \{x \in F^\perp \mid \langle x, y \rangle \geq 0 \text{ for all } y \in P\} \\ &= F^\perp \cap P, \end{aligned}$$

since P is self-dual (5.6(i)). By Corollary 2.10 of [1], $F^\perp \cap P = U(e^\perp)P$. Thus $(F_+^\perp)^\dagger \cap F^\perp = U(e^\perp)P$. But, as is noted on p. 21 of [1], $U(e^\perp) \leq U(e)^\perp$, so $(F_+^\perp)^\dagger \cap F^\perp = U(e^\perp)P \subseteq (U(e)^\perp P)^{c1} = f_+^\perp$ and the proof of (ii) is complete.

5.10 We note in passing that from equation (2.36) on p. 22 of [1], it follows easily that $L(F) = \{a \in F^\perp \mid a = 2e \circ a\}$.

5.11 An idempotent is called *minimal* if there is no nonzero idempotent strictly less than it. We say that an idempotent has *degree* n (or *rank* n) if it can be expressed as the sum of orthogonal minimal idempotents. One can show without much difficulty that the degree of an idempotent is well defined [6, Kapitel III, Satz 7.4]. We say that the algebra itself has degree n if n is the degree of the identity. In the list presented in paragraph 5.1, we note that \mathbf{R} has degree 1, each \mathfrak{S}_n has degree 2 (!), and $\mathfrak{S}_n(\mathbf{K})$ has degree n where \mathbf{K} is either $\mathbf{R}, \mathbf{C}, \mathbf{Q}$ or \mathbf{O} .

5.12 LEMMA. (i) *The group of automorphisms of \mathfrak{A} , $\text{Aut}(\mathfrak{A})$, is a closed subgroup of $O(\mathfrak{A})$ where $O(\mathfrak{A})$ denotes the linear transformations on \mathfrak{A} which are orthogonal with respect to the Killing form.*

(ii) *If \mathfrak{A} is simple, and if $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ are two systems of orthogonal minimal idempotents which each have sum $\mathbf{1}$, then there is an α in $\text{Aut}(\mathfrak{A})$ such that $\alpha(e_i) = f_i$, $i = 1, 2, \dots, n$.*

PROOF. Assertion (i) is proved in Satz 4.5, Kapitel XI, of [6]. Assertion (ii) is proved as Satz 5.5, Kapitel XI, of [6].

5.13 We close out this section by taking a quick look at the cone P associated with an \mathfrak{S}_n , $n \geq 3$. By Lemma 5.5 and definition 5.6, P is the set of squares in \mathfrak{S}_n . Using the notation of paragraph 5.1, we need to decide when a $\beta\mathbf{1} + v$ in \mathfrak{S}_n can be written as $(\alpha\mathbf{1} + u)^2 = (\alpha^2 + \|u\|^2)\mathbf{1} + 2\alpha u$. The first two things which are clear are that β must be nonnegative and $v = 2\alpha u$. Since also $\beta = \alpha^2 + \|u\|^2$, we arrive at the equation $\alpha^4 - \alpha^2\beta + \frac{1}{4}\|v\|^2 = 0$; or, $\alpha^2 = (\beta \pm \sqrt{\beta^2 - \|v\|^2})/2$. But the only time this last equation has real solutions is when $\beta^2 \geq \|v\|^2$ and this is the only constraint. That is, given $\beta \geq 0$ and v (orthogonal to $\mathbf{1} = (0, \dots, 1)$) such that

$\beta \geq \|v\|$, then $\beta \mathbf{1} + v = (\alpha \mathbf{1} + u)^2$ where

$$\alpha = \pm \sqrt{\frac{\beta \pm \sqrt{\beta^2 - \|v\|^2}}{2}}$$

and $u = v/2\alpha$. Thus we see that P is just the forward light cone in \mathbf{R}^n ; i.e., in the usual notation, $P = \{(x_1, \dots, x_n) \mid x_n \geq 0, x_n^2 \geq \sum_{i=1}^{n-1} x_i^2\}$. An element $\alpha \mathbf{1} + u$ of \mathfrak{S}_n is idempotent precisely when $\alpha^2 + \|u\|^2 = \alpha$ and $2\alpha u = u$. If $u = 0$, then we get $\alpha = 0$ or 1 , i.e., the idempotent is 0 or $\mathbf{1}$. But if $u \neq 0$, we find that $\alpha = \frac{1}{2}$ and $\|u\| = \frac{1}{2}$. Thus we may write each idempotent in \mathfrak{S}_n as $\frac{1}{2}\mathbf{1} + \theta/2$ where θ is an element of S^{n-2} (where we identify \mathbf{R}^{n-1} with the vectors orthogonal to $\mathbf{1}$ in \mathbf{R}^n). Clearly $(\frac{1}{2}\mathbf{1} + \theta/2)^\perp = \frac{1}{2}\mathbf{1} - \theta/2$. Thus we have justified algebraically the geometrically obvious fact that each face F of P may be described as the ray $\{\lambda(\frac{1}{2}\mathbf{1} + \theta/2) \mid \lambda \geq 0\}$ for a suitable θ in S^{n-2} and that $F_+^\perp \ominus H(L) = \{\lambda(\frac{1}{2}\mathbf{1} - \theta/2) \mid \lambda \geq 0\}$, the ray in P orthogonal to F .

6. Wiener-Hopf operators on homogeneous self-dual cones. In this section we describe $\mathfrak{B}(P)$ when P is an *irreducible*, homogeneous, self-dual cone; that is, we will assume that P can't be written as the product of lower dimensional cones. In terms of the associated Jordan algebra \mathfrak{A} , to make such an assumption is tantamount to assuming that \mathfrak{A} is simple. The reason we make this assumption is so that we may invoke Lemma 4.12 to assert that for each k less than or equal the degree of \mathfrak{A} , Π_k , the collection of idempotents in \mathfrak{A} of degree k , is a compact homogeneous space under the natural action of $\text{Aut}(\mathfrak{A})$. This assumption provides certain technical simplifications in the arguments, but the reader will easily see that it does not result in any essential loss of generality.

Our analysis complements and extends parts of Berger's and Coburn's investigation [3] as well as their joint work with Koranyi [4].

6.1 Throughout this section, P will denote a fixed, but arbitrary, irreducible, homogeneous, self-dual cone and \mathfrak{A} will be the associated Jordan algebra; n will be the degree of \mathfrak{A} . According to the prescription given in 3.13 coupled with the result of Lemma 5.9, a potential model for the maximal ideal space of the C^* -algebra generated by the functions $1_P * f, f \in L^1(\mathfrak{A})$, is

$$Y = \{(e, a) \mid e \text{ is an idempotent of } \mathfrak{A}, a \in \mathfrak{A}, \text{ and } U(e^\perp)a = a\}.$$

For X we take $\{(e, a) \in Y \mid a \in P\}$ and we let \mathfrak{A} act on Y according to the formula

$$(e, a) \oplus t = (e, a + U(e^\perp)t), \quad (e, a) \in Y, t \in \mathfrak{A}.$$

We write Y_k for $\{(e, a) \in Y \mid e \in \Pi_k\}$ and similarly for X_k . Clearly Y_0 may be identified with \mathfrak{A} and X_0 with P . Accordingly, we extend a function $1_P * f, f \in L^1(\mathfrak{A})$, on \mathfrak{A} to all of Y by the formula

$$\varphi(e, a) = \int_{a + U(e)\mathfrak{A} - P} f(t) dt.$$

We remind the reader that $U(e)\mathfrak{A} - P$ stands for the *closure* of $\{U(e)a - y \mid a \in \mathfrak{A}, y \in P\}$ and that the closure must be taken since in general this set is not closed. We

note, too, that Lemma 5.9 implies that $(U(e)\mathfrak{A} - P) \cap (U(e)\mathfrak{A} + P) = U(e^\perp)^\perp \mathfrak{A}$ and consequently we find the useful fact that for any $a \in \mathfrak{A}$,

$$(1) \quad \varphi(e, U(e^\perp)a) = \int_{a+U(e)\mathfrak{A}-P} f(t) dt.$$

This formula will be used in the following lemma which extends Proposition 3.10 and which is the key to defining the topology on Y (cf. Lemma 4.4).

6.2 LEMMA. *Let $f \in L^1(\mathfrak{A})$ and let φ be the associated function on Y . Let K be a compact subset of \mathfrak{A} and let $\varepsilon > 0$ be given. Then there is a $\lambda \geq 0$ such that for each idempotent e of \mathfrak{A} and for each t in $U(e)P$ with $\lambda e \leq t$,⁵ the inequality*

$$|\varphi(e, U(e^\perp)a) - \varphi(g, U(g^\perp)(a+t))| < \varepsilon.$$

holds for all idempotents $g \leq e$, and all $a \in K$.

In short, λ depends only on f , K , and ε . The lemma asserts, basically, that $\varphi(e, U(e^\perp)a) = \lim_{t \in U(e)\mathfrak{A}} \varphi(g, U(g^\perp)(a+t))$ uniformly in e and $a \in K$.

PROOF. First observe that since the Π_k 's are compact, we need only show that if $e_0 \in \Pi_k$, then there is a neighborhood \mathfrak{U} of e_0 in Π_k and a λ so that the desired conclusion holds for all e in \mathfrak{U} . For any $a \in \mathfrak{A}$, $t \in U(e_0)P$, and $g \leq e_0$ we have

$$\begin{aligned} a+t-P &\subseteq a+t+U(g)\mathfrak{A}-P = U(g^\perp)(a+t) + U(g)\mathfrak{A}-P \\ &\subseteq a+t+U(e_0)\mathfrak{A}-P \\ &= a+U(e_0)\mathfrak{A}-P = U(e_0^\perp)a + U(e_0)\mathfrak{A}-P. \end{aligned}$$

Hence by Proposition 3.10, we may find a λ such that for $t \in U(e_0)P$, $t \geq \lambda e_0$, $\varphi(e_0, U(e_0^\perp)a)$ and $\varphi(g, U(g^\perp)(a+t))$ differ by as little as we please for all $a \in K$ and $g \leq e_0$. Since Π_k is homogeneous, we may find local cross sections to the quotient map mapping $\text{Aut}(\mathfrak{A})$ onto Π_k . Consequently, we may find a neighborhood \mathfrak{U} of e_0 and a continuous map α from \mathfrak{U} into $\text{Aut}(\mathfrak{A})$ such that $\alpha(e)e = e_0$ for all $e \in \mathfrak{U}$. But then for $e \in \mathfrak{U}$ we find that

$$\varphi(e, U(e^\perp)a) = \int_{U(e_0)\mathfrak{A}-P} f(\alpha(e)(t-a)) dt$$

and a similar expression may be obtained for $\varphi(g, U(g^\perp)(a+t))$. Utilizing the compactness of $[\bigcup_{e \in \mathfrak{U}} \alpha(e)K]^{c1}$, the obvious estimates show that we may cut down \mathfrak{U} , if desired, and produce a λ that works for all $e \in \mathfrak{U}$. We omit the details.

6.3 From Lemma 6.2 we are led to define the topology on Y as that which has a basis of open sets of the form

$$\mathcal{V}_{\mathfrak{U}, \emptyset} = \{(g, U(g^\perp)(a + U(e)y)) \mid e \in \mathfrak{U}, g \leq e, a \in \emptyset, y \in \text{Int}(P)\}$$

where \mathfrak{U} is an open subset of some Π_k and where \emptyset is open in \mathfrak{A} . It's worth noting here that $U(e)(\text{Int } P) = \text{Int}(U(e)P)$ (this follows easily from the fact that $\text{Int } P$

⁵From here on we shall simply write \leq for \leq_P and this is the only order with which we will be concerned. This is because unlike the polyhedral case where F_+^\perp need not be contained in P when F is a face, all of the various cones we consider will actually be faces of P and so the orderings on them will be induced by \leq_P .

coincides with the invertible elements in P , cf. Lemma 5.6) and that $\{\lambda e \mid \lambda > 0\}$ is cofinal in the directed set $U(e)\mathfrak{A}$. Consequently it follows immediately from Lemma 6.2 that for each $f \in L^1(\mathfrak{A})$, the associated φ is continuous on Y . Observe, too, that the action of \mathfrak{A} on Y is clearly continuous. Finally note that from the form of the basic open sets, Y_0 is open and dense in Y .

6.4 THEOREM. *With respect to the topology just defined, Y is a locally compact Hausdorff space and the action of \mathfrak{A} on Y is continuous. The map $1_P * f \rightarrow \varphi$, where $f \in L^1(\mathfrak{A})$ and φ is the associated function on Y , extends to an equivariant, *-isomorphism from the C*-algebra A generated by $\{1_P * f \mid f \in L^1(\mathfrak{A})\}$ onto $C_0(Y)$. If σ is the homeomorphism from Y to the maximal ideal space of A which is dual to this isomorphism, and if τ is the imbedding of \mathfrak{A} into the maximal ideal space of A defined in paragraph 3.2, then $\sigma^{-1} \circ \tau$ is simply the identification of \mathfrak{A} with Y_0 , $a \rightarrow (0, a)$. The image of P under $\sigma^{-1} \circ \tau$ is X_0 while the closure of X_0 is X and is compact.*

Thus Y is a model for the maximal ideal space of A and X is a model for the closure of $\tau(P)$.

PROOF. The proof is similar to the proof of Theorem 4.6 and so will be abbreviated at certain points. To begin, we prove directly that X is compact; this will show, too, that Y is locally compact. The proof that X is compact utilizes the result of Theorem 4.6 and Lemma 5.12. Fix a family $\{f_i\}_{i=1}^n$ of minimal, orthogonal idempotents such that $\sum_{i=1}^n f_i = \mathbf{1}$. Let \tilde{P} be the positive orthant in \mathbf{R}^n , let $\{e_i\}_{i=1}^n$ be the standard basis for \mathbf{R}^n , and let Φ_0 be the linear map from \mathbf{R}^n into \mathfrak{A} which carries e_i to f_i . Observe that we may view \mathbf{R}^n as the formally real Jordan algebra which is the direct sum of copies of \mathbf{R} and that when this is done, Φ_0 is a Jordan injection of \mathbf{R}^n into \mathfrak{A} carrying \tilde{P} into P . A face F of \tilde{P} is completely determined by basis elements it contains, F^\perp is spanned by the complementary basis elements, and F_+^\perp is the cone generated by the complementary basis elements. Let $\tilde{Y} = \{(F, y) \mid F \text{ a face of } \tilde{P}, y \in \mathbf{R}^n, U(F^\perp)y = y\}$ be the space constructed from \tilde{P} in §4, give it the topology defined there, and let \tilde{X} be the compact subset $\{(F, x) \in \tilde{Y} \mid x \in F_+^\perp\} = \{(F, x) \in \tilde{Y} \mid x \in \tilde{P}\}$. Now define a map Φ from $\tilde{Y} \times \text{Aut}(\mathfrak{A})$ to Y by the formula

$$\Phi((F, y), \alpha) = \left(\alpha \circ \Phi_0 \left(\sum_{e_i \in F} e_i \right), \alpha \circ \Phi_0(y) \right).$$

By Lemma 5.12, it is easy to see that Φ is surjective and from the definition of the topologies involved, it is also easy to see that Φ is continuous. Since Φ maps $\tilde{X} \times \text{Aut}(\mathfrak{A})$ onto X and since \tilde{X} is compact, X is compact as was to be proved.

The proof that φ belongs to $C_0(Y)$ is the same as that in Theorem 4.6. We show that the collection of such φ separates the points of Y . Suppose that $\varphi(e, a) = \varphi(g, b)$ for all φ . From the integral defining φ and regularity, we conclude that $a + U(e)\mathfrak{A} - P = b + U(g)\mathfrak{A} - P$. From Lemma 5.9 we may conclude from this that $-a + (U(e^\perp)^\perp \mathfrak{A}) \oplus U(e^\perp)P = -b + (U(g^\perp)^\perp \mathfrak{A}) \oplus U(g^\perp)P$ (remember, $U(e^\perp)^\perp$ is larger than $U(e)$). This yields the equation

$$U(e^\perp)P = (a + U(e^\perp)b) + U(e^\perp)((U(g^\perp)^\perp \mathfrak{A}) \oplus U(g^\perp)P).$$

Since $U(e^\perp)P \cap -U(e^\perp)P = \{0\}$, $U(e^\perp)P$ contains no affine subspace of \mathfrak{A} (cf. the proof of 4.6) and so $U(e^\perp)U(g^\perp)^\perp = 0$. Hence $g^\perp \leq e^\perp$; and in a like fashion $e^\perp \leq g^\perp$; i.e., $e = g$. Thus $-a + U(e^\perp)P = -b + U(e^\perp)P$, and since $U(e^\perp)P \cap -U(e^\perp)P = \{0\}$, $a = b$ as well.

Since the rest of the proof is the same as the proof of 4.6, we end it here.

6.5 By Theorem 6.4 the groupoid \mathfrak{G} whose representation theory yields that of $\mathfrak{B}(P)$ is $(Y \times \mathfrak{A})|X$. The Haar system we use is, of course, $\{\lambda^{(e,a)}\}_{(e,a) \in X}$ where $\lambda^{(e,a)}$ is the restriction to \mathfrak{G} of the measure $\delta_{(e,a)} \times \lambda$ and λ denotes Lebesgue measure on \mathfrak{A} . Proposition 2.15 and Theorem 3.7 imply that W , the Wiener-Hopf representation of $C^*(\mathfrak{G})$, is faithful and effects an isomorphism between $C^*(\mathfrak{G})$ and $\mathfrak{B}(P)$. From now on we treat $C^*(\mathfrak{G})$ and $\mathfrak{B}(P)$ interchangeably. As we saw in 6.3, Y_0 is open in Y and so X_0 , which is essentially P , is open in X . Thus X is a regular compactification of P and by Corollary 3.7.2, $\mathfrak{B}(P)$ contains $\mathcal{K}(L^2(P))$. We write $\tilde{X}_k = \bigcup_{j=0}^k X_j$, $k = 0, 1, \dots, n$, obtaining open invariant subsets of X . According to Proposition 2.16, we may identify $C^*(\mathfrak{G}|\tilde{X}_k)$ with a closed, two-sided ideal I_k in $C^*(\mathfrak{G})$ or $\mathfrak{B}(P)$ and I_0 is $\mathcal{K}(L^2(P))$. We write Z_k for $\{(e, c) \in \Pi_k \times \mathfrak{A} \mid U(e^\perp)c = 0\}$ and we note that with the obvious topology, Z_k is a vector bundle over Π_k . The union of the Z_k , $k = 0, 1, \dots, n$, will be denoted by Z .

Motivated by the discussion in 4.8, we are led to the following construction. For each (e, c) in Z and $f \in C_c(\mathfrak{G})$, we define $L^{(e,c)}(f)$ on $L^2(U(e^\perp)P)$ by the formula

$$\begin{aligned} L^{(e,c)}(f)\xi(b) &= \int_{\mathfrak{A}} f(e, b, a) 1_{U(e^\perp)P}(b + U(e^\perp)a) \\ &\quad \times \exp(-i\langle U(e^\perp)^\perp a, c \rangle) \xi(b + U(e^\perp)a) da, \end{aligned}$$

$\xi \in L^2(U(e^\perp)P)$. Here, of course, $U(e^\perp)P$ is given Lebesgue measure on $U(e^\perp)\mathfrak{A}$, and $\langle \cdot, \cdot \rangle$ denotes the Killing form on \mathfrak{A} (5.2). A straightforward calculation shows that $L^{(e,c)}$ extends to a representation of $C^*(\mathfrak{G})$ on $L^2(U(e^\perp)P)$. Observe that $L^{(0,0)} = W$.

To understand $L^{(e,c)}$ a bit better when $(e,c) \neq (0,0)$, observe first that the orbits in $\mathfrak{G}^0 = X$ are determined by the idempotents in \mathfrak{A} , the orbit of e being $\{(e, a) \mid a \in U(e^\perp)P\}$. By definition of the topology on Y , the closure $X(e)$ of this orbit is simply $\{(g, b) \in X \mid e \leq g\}$. On the other hand, by definition of $L^{(e,c)}$, $L^{(e,c)}(f) = 0$ if $f \in C_c(\mathfrak{G})$ is supported on $\mathfrak{G} \setminus (X \setminus X(e))$. So, by Proposition 2.16, we may view $L^{(e,c)}$ as a representation of $C^*(\mathfrak{G}|\mathcal{K}(X(e)))$. But $\mathfrak{G}|\mathcal{K}(X(e)) = \{(g, a, b) \mid e \leq g, a \in U(g^\perp)P, a + U(g^\perp)b \in U(g^\perp)P\}$. Now let $\mathfrak{G}(e) = \{(g, a, b) \in \mathfrak{G}|\mathcal{K}(X(e)) \mid b \in U(e^\perp)\mathfrak{A}\}$ with the subspace topology and obvious Haar system and observe that $\mathfrak{G}(e)$ is essentially just the groupoid associated with the Wiener-Hopf operators on $U(e^\perp)P$ regarded as a cone in $U(e^\perp)\mathfrak{A}$. If \mathfrak{A} is decomposed as $U(e^\perp)\mathfrak{A} \oplus U(e^\perp)^\perp\mathfrak{A}$, then the map $(g, a, (b_1 \oplus b_2)) \rightarrow ((g, a, b_1), b_2)$ from $\mathfrak{G}|\mathcal{K}(X(e))$ to $\mathfrak{G}(e) \times U(e^\perp)^\perp\mathfrak{A}$, where $U(e^\perp)^\perp\mathfrak{A}$ is given Lebesgue measure, is a topological isomorphism. This isomorphism implements a C^* -isomorphism between $C^*(\mathfrak{G}|\mathcal{K}(X(e)))$ and $C^*(\mathfrak{G}(e)) \otimes C^*(U(e^\perp)^\perp\mathfrak{A})$ and carries $L^{(e,c)}$ into the tensor product of the Wiener-Hopf representation on $C^*(\mathfrak{G}(e))$ with the character of $C^*(U(e^\perp)^\perp\mathfrak{A})$ determined by c . It follows that $L^{(e,c)}$ is irreducible; moreover, it follows that if $(f, d) \in Z$ is different

from (e, c) , then $L^{(e,c)}$ and $L^{(f,d)}$ have different kernels and so are inequivalent. The following theorem summarizes this discussion and completes our analysis of $\mathfrak{B}(P)$.

6.6 THEOREM. (i) *The C*-algebra $\mathfrak{B}(P)$ is postliminal, and the ideals I_k , $k = 0, 1, 2, \dots, n$, form a composition series for $\mathfrak{B}(P)$.*

(ii) *For each $(e, c) \in Z$, $L^{(e,c)}$ is irreducible; no two $L^{(e,c)}$ are equivalent; and every irreducible representation of $\mathfrak{B}(P)$ is equivalent to some $L^{(e,c)}$. Thus as a set the spectrum of $\mathfrak{B}(P)$ may be taken to be Z .*

(iii) *For each $k = 0, 2, \dots, n-1$, I_k/I_{k-1} is isomorphic to $C_0(Z_k) \otimes \mathcal{K}$, while I_n/I_{n-1} is isomorphic to $C_0(\mathfrak{A})$ ($I_{-1} = \{0\}$ by convention).*

(iv) *The topology on Z has a base consisting of sets of the form $\emptyset \cup (\cup_{j=0}^{k-1} Z_j)$ where \emptyset is open in Z_k .*

PROOF. The proof really revolves around proving (iii). Indeed, since the ideals I_k , $k = 0, 1, \dots, n$, form a composition series for $\mathfrak{B}(P)$ and since (iii) implies that the consecutive quotients are liminal, the fact that $\mathfrak{B}(P)$ is postliminal follows from [11, 4.3.4]. Also, we already know that the $L^{(e,c)}$ are irreducible and inequivalent for different points (e, c) in Z . So to prove (ii), we need only show that the $L^{(e,c)}$ exhaust the spectrum of $\mathfrak{B}(P)$. This will fall out of the proof of (iii). Finally since $I_0 = \mathcal{K}(L^2(P))$, its spectrum is the singleton determined by the Wiener-Hopf representation W and may therefore be viewed as a dense open set of the spectrum of $\mathfrak{B}(P)$. Consequently the spectrum of each I_k may be viewed as a dense open set in the spectrum of $\mathfrak{B}(P)$. But then (iv) follows from (iii) which identifies the spectrum of I_k/I_{k-1} with Z_k .

To prove (iii), we first prove that I_k/I_{k-1} , $k = 1, 2, \dots, n-1$, is isomorphic to the C*-algebra defined by a locally trivial, continuous field $((A(e, c)), \Theta)_{(e,c) \in Z_k}$ of C*-algebras over Z_k where the fiber algebras $A(e, c)$ are isomorphic to \mathcal{K} . (We follow the notation and terminology of [11, Chapter 10].) Then we produce a continuous field of separable, infinite dimensional Hilbert spaces $(\mathcal{H}(e, c), \Gamma)_{(e,c) \in Z_k}$ such that Θ contains the fields $\theta_{x,y}$, $x, y \in \Gamma$, where for (e, c) , $\theta_{x,y}(e, c)$ is the rank one operator on $\mathcal{H}(e, c)$ defined by the formula $\theta_{x,y}(e, c)(\xi) = (\xi, y(e, c))x(e, c)$. It will follow, then, by Lemma 10.5.3 of [11], a special case of the Stone-Weierstrass theorem for C*-algebras, that $(A(e, c), \Theta)_{(e,c) \in Z_k}$ is isomorphic to the field determined by $(\mathcal{H}(e, c), \Gamma)_{(e,c) \in Z_k}$. Since Z_k is obviously paracompact and finite dimensional, the field of Hilbert spaces is trivial by Lemma 10.8.7 of [11]. Hence the C*-algebra determined by $(\mathcal{H}(e, c), \Gamma)_{(e,c) \in Z_k}$ is isomorphic to $C_0(Z_k) \otimes \mathcal{K}$. This will prove that I_k/I_{k-1} is isomorphic to $C_0(Z_k) \otimes \mathcal{K}$, as desired.

To prove that I_k/I_{k-1} is defined by a locally trivial continuous field of C*-algebras, we begin by recalling that by Proposition 2.16, I_k/I_{k-1} is isomorphic to $C^*(\mathfrak{G} | X_k)$ and by noting that $\mathfrak{G} | X_k = \{(e, a, b) \in \mathfrak{G} | e \in \Pi_k\}$. For each open set \mathcal{Q} in Π_k , let $\tilde{\mathcal{Q}} = \{(e, a) \in X_k | e \in \mathcal{Q}\}$. Then each $\tilde{\mathcal{Q}}$ is an open invariant subset of $X_k = (\mathfrak{G} | X_k)^0$. So, by Proposition 2.16 again, we may regard $C^*(\mathfrak{G} | \tilde{\mathcal{Q}})$ as an ideal in $C^*(\mathfrak{G} | X_k)$. Fix $e_0 \in \Pi_k$ and choose a neighborhood \mathcal{Q} of e_0 on which there is a continuous map α into $\text{Aut}(\mathfrak{A})$ such that $\alpha(e)e = e_0$ for all $e \in \mathcal{Q}$. Let $\mathfrak{F}(e_0)$ be the reduction to $U(e_0^\perp)P$ of the transformation group groupoid determined by $U(e_0^\perp)\mathfrak{A}$

acting by translation on $U(e_0^\perp)\mathfrak{A}$ and give $\mathfrak{H}(e_0)$ the usual Haar system (cf. 2.5). Thus

$$\mathfrak{H}(e_0) = \{(a, b) \in U(e_0^\perp)P \times U(e_0^\perp)\mathfrak{A} \mid a + b \in U(e_0^\perp)P\}.$$

The argument of paragraph 2.3 shows that $\mathfrak{H}(e_0)$ is topologically isomorphic to the trivial groupoid on $U(e_0^\perp)P$ and so $C^*(\mathfrak{H}(e_0))$ is isomorphic to \mathcal{K} (cf. 2.17.1). Now let $\mathfrak{H}_{\mathfrak{U}}$ be the product, $\mathfrak{U} \times \mathfrak{H}(e_0) \times U(e_0^\perp)^\perp \mathfrak{A}$, of the cotrivial groupoid \mathfrak{U} (2.2.2 and 2.5), $\mathfrak{H}(e_0)$, and the group $U(e_0^\perp)^\perp \mathfrak{A}$, and give $\mathfrak{H}_{\mathfrak{U}}$ the product Haar system. Then $C^*(\mathfrak{H}_{\mathfrak{U}})$ is isomorphic to $C^*(\mathfrak{U}) \otimes C^*(\mathfrak{H}(e_0)) \otimes C^*(U(e_0^\perp)^\perp \mathfrak{A})$ which in turn is isomorphic to $C_0(\mathfrak{U} \times U(e_0^\perp)^\perp \mathfrak{A}) \otimes \mathcal{K}$. But $\mathfrak{U} \times U(e_0^\perp)^\perp \mathfrak{A}$ is homeomorphic to the open set in Z_k , $\mathcal{O}(\mathfrak{U}) = \{(e, c) \in Z_k \mid e \in \mathfrak{U}\}$, under the map $(e, c) \rightarrow (e, \alpha(e)^{-1}c)$ and so, finally, $C^*(\mathfrak{H}_{\mathfrak{U}})$ is isomorphic to $C_0(\mathcal{O}(\mathfrak{U})) \otimes \mathcal{K}$. On the other hand, the map Φ from $\mathfrak{G} \mid \tilde{\mathfrak{U}}$ to $\mathfrak{H}_{\mathfrak{U}}$ defined by the formula

$$\Phi(e, a, b) = (e, (\alpha(e)a, \alpha(e)U(e^\perp)b), \alpha(e)U(e^\perp)^\perp b)$$

is a topological isomorphism from $\mathfrak{G} \mid \tilde{\mathfrak{U}}$ onto $\mathfrak{H}_{\mathfrak{U}}$ and so implements an isomorphism between $C^*(\mathfrak{G} \mid \tilde{\mathfrak{U}})$ and $C_0(\mathcal{O}(\mathfrak{U})) \otimes \mathcal{K}$. In particular, then, we see that the spectrum of $C^*(\mathfrak{G} \mid \tilde{\mathfrak{U}})$ is (homeomorphic to) $\mathcal{O}(\mathfrak{U})$. Notice that if $(e, c) \in \mathcal{O}(\mathfrak{U})$, then the formula defining $L^{(e, c)}$ makes perfectly good sense for $f \in C_c(\mathfrak{G} \mid \tilde{\mathfrak{U}})$ and so we may view $L^{(e, c)}$ as defining a representation $L_1^{(e, c)}$ of $C^*(\mathfrak{G} \mid \tilde{\mathfrak{U}})$. The analysis just completed shows simply that $L_1^{(e, c)}$ is (equivalent to) the tensor product of the character of $C_0(\mathcal{O}(\mathfrak{U}))$ determined by (e, c) and the representation $\text{Ind } \delta_0$ of $C^*(\mathfrak{H}(e_0))$. Conversely every irreducible representation of $C^*(\mathfrak{G} \mid \mathfrak{U})$ is equivalent to a unique $L^{(e, c)}$ for some $(e, c) \in \mathcal{O}(\mathfrak{U})$. Cover Π_k by a finite family $\mathfrak{U}_1, \dots, \mathfrak{U}_p$ of open sets of the kind we have been discussing, and observe that $C^*(\mathfrak{G} \mid X_k) = C^*(\mathfrak{G} \mid \mathfrak{U}_1) + \dots + C^*(\mathfrak{G} \mid \mathfrak{U}_p)$. (The sum is not direct, it is simply meant to state that every element of $C^*(\mathfrak{G} \mid X_k)$ may be written as a sum of elements extracted from the $C^*(\mathfrak{G} \mid \mathfrak{U}_j)$.) To see this, choose a partition of unity, $\{h_j\}_{j=1}^p$, on X_k subordinate to $\{\mathfrak{U}_j\}_{j=1}^p$. Then Proposition 1.14 in Chapter II of [25] implies that we may regard each h_j as a multiplier of $C^*(\mathfrak{G} \mid X_k)$. (Actually, the proposition is stated for compactly supported functions on the unit space of a groupoid, but the proof extends without change to cover bounded continuous functions.) So if $f \in C^*(\mathfrak{G} \mid X_k)$, then $f = \sum M(h_j)f$ and $M(h_j)f$ belongs to $C^*(\mathfrak{G} \mid \mathfrak{U}_j)$, $j = 1, \dots, p$. Suppose now that L is an irreducible representation of $C^*(\mathfrak{G}) \cong \mathfrak{B}(P)$ and assume that k is the first integer such that L does not annihilate $C^*(\mathfrak{G} \mid \tilde{X}_k)$. Then by [11, 2.10.4], L is uniquely determined by its restriction to $C^*(\mathfrak{G} \mid \tilde{X}_k)$. Since L annihilates $C^*(\mathfrak{G} \mid \tilde{X}_{k-1})$ we may pass to the quotient and regard L as an irreducible representation of $C^*(\mathfrak{G} \mid X_k)$. But then, since $C^*(\mathfrak{G} \mid X_k) = C^*(\mathfrak{G} \mid \mathfrak{U}_1) + \dots + C^*(\mathfrak{G} \mid \mathfrak{U}_p)$, L does not annihilate some $C^*(\mathfrak{G} \mid \mathfrak{U}_j)$. From what we've seen, the restriction of L to $C^*(\mathfrak{G} \mid \mathfrak{U}_j)$ is equivalent to a unique $L_1^{(e, c)}$ for some $(e, c) \in \mathcal{O}(\mathfrak{U}_j)$. It follows that L itself on $C^*(\mathfrak{G})$ is equivalent to $L^{(e, c)}$. Thus as a set the spectrum of $C^*(\mathfrak{G} \mid X_k)$ may be identified with Z_k while that of $C^*(\mathfrak{G})$ may be identified with Z . However, since the spectrum of $C^*(\mathfrak{G} \mid \mathfrak{U}_j)$ is homeomorphic to $\mathcal{O}(\mathfrak{U}_j)$, it follows from [11, 3.2.2] that the topology on the spectrum of $C^*(\mathfrak{G} \mid X_k)$ coincides with that on Z_k . Finally, since $C^*(\mathfrak{G} \mid \mathfrak{U}_j)$ is isomorphic to $C_0(\mathcal{O}(\mathfrak{U}_j)) \otimes \mathcal{K}$,

we conclude that $C^*(\mathcal{G} | X_k)$ is isomorphic to the C^* -algebras defined by a locally trivial continuous field over Z_k with fiber algebras isomorphic to \mathbb{R} . This completes the first step in the proof of (iii).

Next we produce $(\mathcal{H}(e, c), \Gamma)_{(e, c) \in Z_k}$. Of course we choose $L^2(U(e^\perp)P)$, the space of $L^{(e, c)}$, for $\mathcal{H}(e, c)$. To produce Γ , we need only exhibit a linear subspace Γ' of $\Pi_{(e, c) \in Z_k} \mathcal{H}(e, c)$ such that for each (e, c) , $\{x(e, c) | x \in \Gamma'\}$ is dense in $\mathcal{H}(e, c)$ and such that the function $(e, c) \rightarrow \|x(e, c)\|$ is continuous (cf. [11, Proposition 10.2.3]). For each $e \in \Pi_k$, we decompose \mathfrak{A} into $U(e^\perp)\mathfrak{A} \oplus U(e^\perp)^\perp \mathfrak{A}$ with a corresponding decomposition of Lebesgue measure on \mathfrak{A} . If f is a compactly supported continuous function on \mathfrak{A} , then we define the field $x(e, c)$ determined by f through the formula

$$(2) \quad x(e, c)(\alpha) = \int_{U(e^\perp)^\perp \mathfrak{A}} f(\alpha, b) \exp(-i\langle b, c \rangle) db.$$

By definition, we take for Γ' the collection of all such fields obtained by letting f run over $C_c(\mathfrak{A})$. It is a routine matter to check that Γ' has all the desired properties. In particular, we note that the continuity of the map $(e, c) \rightarrow \|x(e, c)\|$ follows easily from the existence of local cross sections to the quotient map from $\text{Aut}(\mathfrak{A})$ to Π_k . We omit the tedious details and proceed to analyze the continuous field $(\mathcal{Q}(e, c), \Theta_0)$ of elementary C^* -algebras associated with $(\mathcal{H}(e, c), \Gamma)_{(e, c) \in Z_k}$.

To this end, let $x = x(e, c)$ and $y = y(e, c)$ be the fields in Γ determined by functions f and g , respectively, using equation (2). Then the rank one operator field $\theta_{x, y} = \theta_{x, y}(e, c)$ determined by x and y is given by the following formula in which $\xi \in \mathcal{H}(e, c) = L^2(U(e^\perp)P)$,

$$f * {}_1 \tilde{g}(b, \alpha, a) = \int_{U(e^\perp)^\perp \mathfrak{A}} f(\beta, c) \tilde{g}(\alpha, a - c) dc,$$

and $\tilde{g}(\alpha, a) = \overline{g(\alpha, -a)}$.

$$\begin{aligned} \theta_{x, y}(e, c)\xi(\beta) &= x(e, c)(\beta)[(\xi, y(e, c))] \\ &= \int_{U(e^\perp)P} x(e, c)(\beta) \overline{y(e, c)(\alpha)} \xi(\alpha) d\alpha \\ (3) \quad &= \int_{U(e^\perp)P} \left[\int_{U(e^\perp)^\perp \mathfrak{A}} f(\beta, a) \exp(-i\langle a, c \rangle) da \right] \\ &\quad \times \left[\int_{U(e^\perp)^\perp \mathfrak{A}} \overline{g(\alpha, b)} \exp(-i\langle -b, c \rangle) db \right] \xi(\alpha) d\alpha \\ &= \int_{U(e^\perp)P} \left[\int_{U(e^\perp)^\perp \mathfrak{A}} f * {}_1 \tilde{g}(\beta, \alpha, a) \exp(-i\langle a, c \rangle) da \right] \xi(\alpha) d\alpha. \end{aligned}$$

The algebra of continuous fields generated by the $\theta_{x, y}$ constitutes a total subset in the desired Θ_0 , and of course $\mathcal{Q}(e, c) = \mathcal{K}(\mathcal{H}(e, c))$.

To complete the proof that I_k/I_{k-1} is isomorphic to $C_0(Z_k) \otimes \mathcal{K}$, $k = 0, 1, \dots, n - 1$, all that remains is to show that if $(A(e, c), \Theta)_{(e, c) \in Z_k}$ is the continuous field associated with I_k/I_{k-1} above, then $\Theta_0 \subseteq \Theta$. For this, it suffices to make Θ explicit and to show that $\theta_{x, y}$ belongs to Θ for every x and y determined by compactly

supported continuous functions through equation (2). In the formula defining $L^{(e,c)}(f)\xi$ where $f \in C_c(\mathbb{G} | X_k)$ and $\xi \in \mathfrak{H}(e, c) = L^2(U(e^\perp)P)$, it is convenient to split $\mathfrak{A} = U(e^\perp)\mathfrak{A} \oplus U(e^\perp)^\perp \mathfrak{A}$ and to write

$$\begin{aligned} L^{(e,c)}(f)\xi(b) &= \int_{U(e^\perp)\mathfrak{A}} \int_{U(e^\perp)^\perp \mathfrak{A}} f(e, \beta, \alpha, a) 1_{U(e^\perp)P} \\ &\quad \times (\beta + \alpha) \exp(-i\langle a, c \rangle) \xi(\beta + \alpha) da d\alpha. \end{aligned}$$

In this formula we change variables ($\alpha \rightarrow \alpha - \beta$) and write $k(e, \beta, \alpha, a)$ for $f(e, \beta, \alpha - \beta, a)$. The formula for $L^{(e,c)}(f)\xi(\beta)$, then, becomes

$$(4) \quad \int_{U(e^\perp)P} \int_{U(e^\perp)^\perp \mathfrak{A}} k(e, \beta, \alpha, a) \exp(-i\langle a, c \rangle) \xi(\alpha) da d\alpha$$

where k is a certain compactly supported, continuous function on $E_k = \{(e, \beta, \alpha, a) | e \in \Pi_k, \beta \in U(e^\perp)P, \alpha \in U(e^\perp)^\perp \mathfrak{A}, \text{ and } a \in U(e^\perp)^\perp \mathfrak{A}\}$. But actually, every compactly supported, continuous function on this set arises from some f in $C_c(\mathbb{G} | X_k)$. Indeed, given k , simply set $f(e, \beta, \alpha, a) = k(e, \beta, \beta + \alpha, a)$. Thus we see that Θ has a total subset consisting of all operator fields determined by the functions k in $C_c(E_k)$. However, equation (3) makes it evident that $\theta_{x,y}$ is such an operator field. (The parameter e is not exhibited explicitly in the kernel $f * {}_1\tilde{g}$, however the kernel does in fact depend upon it and $f * {}_1\tilde{g}$ may be viewed as a function in $C_c(E_k)$.) Thus we see that $\Theta_0 \subseteq \Theta$ and this completes the proof that I_k/I_{k-1} is isomorphic to $C_0(Z_k) \otimes \mathfrak{K}$, $k = 0, 1, \dots, n-1$.

To finish the proof of (iii) we need only remark that I_n/I_{n-1} is isomorphic to $C^*(\mathbb{G} | X_n)$. Since X_n is a point and the isotropy group of that point is \mathfrak{A} , we conclude that I_n/I_{n-1} is isomorphic to $C_0(\mathfrak{A})$. With this, the proof of (iii) and of the theorem is complete.

6.6.1 COROLLARY. *The commutator ideal of $\mathfrak{B}(P)$ is I_{n-1} and the quotient $\mathfrak{B}(P)/I_{n-1}$ is isomorphic to $C_0(\mathfrak{A})$.*

6.6.2 COROLLARY. *The C^* -algebra $\mathfrak{B}(P)$ is solvable of length n .*

PROOF. The proof is identical with the proof of Corollary 4.7.2.

6.7 We conclude with a few remarks about $\mathfrak{B}(P)$ when P is the forward light cone in \mathbb{R}^n , $n \geq 3$. Recall from paragraph 5.13 that the Jordan algebra associated with P is \mathfrak{S}_n and that \mathfrak{S}_n has degree 2. Consequently, by Theorem 6.6, $\mathfrak{B}(P)$ has a composition series of length 2, $\{0\} \subseteq I_0 \subseteq I_1 \subseteq I_2 = \mathfrak{B}(P)$, with $I_0 = \mathfrak{K}$, I_2/I_1 isomorphic to $C_0(\mathbb{R}^n)$, and I_1/I_0 isomorphic to $C_0(Z_1) \otimes \mathfrak{K}$. But from the definition of the product on \mathfrak{S}_n (5.1) and from the definition of the quadratic representation (5.2), straightforward calculation shows that for $e \in I_1$, $U(e)$ is simply the orthogonal projection onto the space spanned by e . Accordingly, we find that $Z_1 = \{(e, c) | e \in I_1, e^\perp \cdot c = 0\}$. Now recall that as an n -tuple, an idempotent e may be written as $(\theta/2, \frac{1}{2})$ where θ is a vector in S^{n-2} and that $e^\perp = (-\theta/2, \frac{1}{2})$. It follows that the map from $S^{n-2} \times \mathbb{R}^{n-1}$ to Z_1 defined by the formula

$$(\theta, a) \rightarrow \left(\left(\frac{\theta}{2}, \frac{1}{2} \right), (a, \theta \cdot a) \right)$$

is a homeomorphism. Thus, in fact, I_1/I_0 is isomorphic to $C_0(S^{n-2} \times \mathbf{R}^{n-1}) \otimes \mathcal{K}$ as asserted in the introduction. Finally we note that an analysis similar to that discussed in the second remark of 4.8 shows that when P is the forward light cone, $\mathfrak{B}(P)/I_0$ may be imbedded canonically into $\mathfrak{B}(U(e_0)\mathfrak{S}_n + P) \otimes C(S^{n-2})$ where e_0 is some fixed idempotent in I_1 . But by 5.9 coupled with 3.7.1, $\mathfrak{B}(U(e_0)\mathfrak{S}_n + P)$ is isomorphic to $\mathfrak{B}(U(e_0^\perp)P) \otimes C_0(\mathbf{R}^{n-1})$. Thus $\mathfrak{B}(P)/I_0$ is isomorphic to $\mathfrak{B}(U(e_0^\perp)P) \otimes C_0(S^{n-2} \times \mathbf{R}^{n-1})$. Since $\mathfrak{B}(U(e_0^\perp)P)$ is really just the algebra of Wiener-Hopf operators on $[0, \infty)$, we see that the problem of deciding when a Wiener-Hopf operator in $\mathfrak{B}(P)$ is Fredholm reduces to the problem of deciding when a certain continuous field of one-dimensional Wiener-Hopf operators is invertible.

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